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Constrained Willmore Surfaces

Symmetries of a Möbius Invariant Integrable System

Based on the author's PhD Thesis

To my parents

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Preface. This work is based on the PhD Thesis with the same title submitted to the University of Bath, Department of Mathematical Sciences, on September 15, 2008 and defended viva voce on the (rather snowy) 2$^{\text{nd}}$ of February 2009. The contents of the Thesis available from the University of Bath has been preserved, with the exception of Sections 8.2.3 and 8.2.4, where the original study of constrained Willmore spectral deformation and Bäcklund transformation of CMC surfaces has been extended to a general multiplier. This version has benefited from the rephrasing or reformulation of parts of some sections.

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Abstract. This work is dedicated to the study of the Möbius invariant class of constrained Willmore surfaces and its symmetries. We define a spectral deformation by the action of a loop of flat metric connections; *Bäcklund transformations*, by applying a dressing action; and, in 4-space, *Darboux transformations*, based on the solution of a Riccati equation. We establish a permutability between spectral deformation and Bäcklund transformation and prove that non-trivial Darboux transformation of constrained Willmore surfaces in 4-space can be obtained as a particular case of Bäcklund transformation. All these transformations corresponding to the zero multiplier preserve the class of Willmore surfaces. We verify that, for special choices of parameters, both spectral deformation and Bäcklund transformation preserve the class of constrained Willmore surfaces admitting a *conserved quantity*, and, in particular, the class of CMC surfaces in 3-dimensional space-form.

Contents

Introduction

Among the classes of Riemannian submanifolds, there is that of Willmore surfaces, named after T. Willmore [**60**] (1965), although the topic was mentioned by W. Blaschke [**4**] (1929) and by G. Thomsen [**55**] (1923). Early in the nineteenth century, S. Germain [**28**], [**29**] studied elastic surfaces. On her pioneering analysis, she claimed that the elastic force of a thin plate is proportional to its mean curvature. Since then, the mean curvature remains a key concept in theory of elasticity. In modern literature on the elasticity of membranes (see, for example, [**37**] and [**40**]), a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature is considered the elastic energy of a membrane. By neglecting the total mean curvature (by physical considerations) and having in consideration that the total Gaussian curvature of compact orientable Riemannian surfaces without boundary is a topological invariant, T. Willmore defined the Willmore energy of a compact oriented Riemannian surface, without boundary, isometrically immersed in $\mathbb{R}^3$, to be

$$\mathcal{W} = \int H^2 dA.$$

The Willmore functional "extends" to isometric immersions of compact oriented Riemannian surfaces in Riemannian manifolds by means of half of the total squared norm of the trace-free part of the second fundamental form, which, in fact, amongst surfaces in $\mathbb{R}^3$, differs from $\mathcal{W}$ by the total Gaussian curvature, but still shares then the critical points with $\mathcal{W}$. Willmore surfaces are the extremals of the Willmore functional - just like harmonic maps are the extremals of the energy functional.

Conformal invariance motivates us to move from Riemannian to Möbius geometry. Our study is a study of geometrical aspects that are invariant under Möbius transformations, with the exception of the study of constant mean curvature surfaces, in Sections 7.2.1 and 8.2. This work restricts to the study of surfaces conformally immersed in n-dimensional space-forms with $n \geq 3$. It starts with a Möbius description of space-forms in the projectivized light-cone, following [**10**]. Such description is based on the model of the conformal n-sphere on the projective space $\mathbb{P}(\mathcal{L})$ of the light-cone $\mathcal{L}$ of $\mathbb{R}^{n+1,1}$,

$$S^n \cong \mathbb{P}(\mathcal{L}),$$

due to Darboux [**21**], which, in particular, yields a conformal description of Euclidean n-spaces and hyperbolic n-spaces as submanifolds of $\mathbb{P}(\mathcal{L})$. We approach then a surface conformally immersed in n-space as a null line subbundle Λ of $\underline{\mathbb{R}}^{n+1,1} = M \times \mathbb{R}^{n+1,1}$ defining an immersion

$$\Lambda : M \to \mathbb{P}(\mathcal{L})$$

of an oriented surface M, which we provide with the conformal structure induced by Λ, into the projectivized light-cone. The realization of all space-forms as submanifolds of the projectivized light-cone arises from the realization, cf. [**10**], of all n-dimensional space-forms as connected components of conic sections

$$S_{v_\infty} := \{v \in \mathcal{L} : (v, v_\infty) = -1\}$$

of the light-cone, with $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero. S_{v_∞} inherits a positive definite metric of constant sectional curvature $-(v_\infty, v_\infty)$ from $\mathbb{R}^{n+1,1}$ and is either a copy of a sphere, a copy of Euclidean space or two copies of hyperbolic space, according to the sign of (v_∞, v_∞). For each v_∞, the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defines a diffeomorphism

$$\pi|_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L}) \backslash \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp). \tag{0.1}$$

We provide $\mathbb{P}(\mathcal{L})$ with the conformal structure of the metric induced by $\pi|_{S_{v_\infty}}$, fixing v_∞ time-like, independently of the choice of v_∞, and, in this way, identify $\mathbb{P}(\mathcal{L})$ with the conformal n-sphere and make each diffeomorphism (0.1) - for a general v_∞, not necessarily time-like - into a conformal diffeomorphism.

In this work, we restrict to surfaces Λ in S^n which are not contained in any subsphere of S^n. Such a surface Λ defines a surface in any given space-form, by means of a lift, whose study is Möbius equivalent to the study of Λ and which will often be considered. Namely, given $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero, we have, locally, $(\sigma, v_\infty) \neq 0$, and Λ defines then a local immersion

$$\sigma_\infty := (\pi|_{S_{v_\infty}})^{-1} \circ \Lambda = \frac{-1}{(\sigma, v_\infty)}\,\sigma : M \to S_{v_\infty},$$

of M into the space-form S_{v_∞}.

Having presented our setup, in Chapter 2, we introduce, following [**14**], the central sphere congruence, a fundamental construction of Möbius invariant surface geometry which will be basic to our study of surfaces. The concept has its origins in the nineteenth century with the introduction of the mean curvature sphere of a surface at a point, by S. Germain [**30**]. By the turn of the century, the family of the mean curvature spheres of a surface was known as the central sphere congruence, cf. W. Blaschke [**4**]. Nowadays, after R. Bryant's paper [**9**], it goes as well by the name conformal Gauss map. The central sphere congruence of a surface in n-space,

$$S : M \to Gr_{(3,1)}(\mathbb{R}^{n+1,1}),$$

defines a decomposition

$$d = \mathcal{D} + \mathcal{N}, \tag{0.2}$$

of the trivial flat connection on $\underline{\mathbb{R}}^{n+1,1}$ into the sum of a connection $\mathcal{D}$, with respect to which S and $S^\perp$ are parallel, and a 1-form $\mathcal{N}$ with values in $S \wedge S^\perp$. Explicitly,

$$\mathcal{D} := \nabla^S + \nabla^{S^\perp}, \ \ \mathcal{N} := d - \mathcal{D},$$

for ∇^S and $\nabla^{S^\perp}$ the connections induced by d on S and $S^\perp$, respectively. Under the standard identification

$$S^*TGr_{(3,1)}(\mathbb{R}^{n+1,1}) \cong \mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp,$$

of bundles provided with a metric and a connection, we have

$$dS = \mathcal{N}, \tag{0.3}$$

which establishes a characterization of the harmonicity of S by

$$d^{\mathcal{D}} * \mathcal{N} = 0.$$

Chapter 4 is dedicated to the class of Willmore surfaces in space-forms and its link to the class of harmonic maps into Grassmannian manifolds via the central sphere congruence. W. Blaschke [**4**] established the Möbius invariance of the Willmore energy of a surface in spherical 3-space. B.-Y. Chen [**18**] generalized it to surfaces in constant curvature Riemnannian manifolds. We present a manifestly conformally invariant formulation of the Willmore energy of a surface Λ in n-dimensional space-form,

$$\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (\mathcal{N} \wedge *\mathcal{N}),$$

following the definition of energy of the mean curvature sphere congruence of a surface in spherical 4-space, presented in [**12**]. The class of Willmore surfaces in n-space is then established as invariant under the group of Möbius transformations of S^n. As immediately established by (0.3), and already known to Blaschke [**4**] for the particular case of spherical 3-space, the Willmore energy of a surface in a space-form coincides with the energy of its central sphere congruence. Furthermore, a result by Blaschke [**4**] (for $n = 3$) and N. Ejiri [**27**] (for general n) characterizes Willmore surfaces in spherical n-space by the harmonicity of the central sphere congruence. Via this characterization, the class of Willmore surfaces in space-forms is associated to a class of harmonic maps into Grassmannians. This will enable us to apply to this class of surfaces the well-developed integrable systems theory of harmonic maps into Grassmannian manifolds, with a spectral deformation and Bäcklund transformations, cf. [**54**] and [**56**].

In many occasions throughout this work, we use an interpretation of loop group theory by F. Burstall and D. Calderbank [**11**] and produce transformations of surfaces

by the action of loops of flat metric connections. Specifically, by replacing the trivial flat connection by another flat metric connection $\tilde{d}$ on $\underline{\mathbb{R}}^{n+1,1}$, we transform (in certain cases) a surface $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ into a *$\tilde{d}$-surface* $\tilde{\Lambda}$, or, equivalently, into another surface $\tilde{\phi}\Lambda$, defined, up to a Möbius transformation, for

$$\tilde{\phi} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$$

an isomorphism of bundles provided with a metric and a connection. Many will be the examples in this work of such transformations preserving the geometrical aspects of a class, i.e., establishing symmetries of integrable systems. Symmetries of integrable systems will arise from other constructions, as well. Throughout this work, by transformation/deformation of a class of surfaces shall be understood a symmetry of the system, i.e., a transformation/deformation of the surfaces in the class into new ones (possibly isomorphic) still in the class.

Chapter 3 is introductory of the idea of a surface under change of flat metric connection. A first example, due to F. Burstall and D. Calderbank [**11**], of a symmetry of an integrable system arising from the action of a loop of flat metric connections establishes the class of Willmore surfaces in space-forms as an integrable system with a spectral deformation, a fact that was already known to F. Burstall et al. [**14**]. According to K. Uhlenbeck [**56**], the harmonicity of S is characterized by the flatness of the real metric connection $d^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1}$ on $(\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$, for each $\lambda \in S^1$. The action of this loop of curvature-free connections defines a S^1-deformation of S into harmonic maps, which, as we verify, is the family of central sphere congruences corresponding to the S^1-deformation of Λ defined by the action of the loop. The characterization of Willmore surfaces in space-forms in terms of the harmonicity of the central sphere congruence gives rise, in this way, to a spectral deformation of Willmore surfaces. This deformation coincides, up to reparametrization, with the one presented in [**14**].

Bäcklund transformations of Willmore surfaces will arise from a more complex construction, following the work of C.-L. Terng and K. Uhlenbeck [**54**].

In Chapter 5, we introduce constrained Willmore surfaces, the generalization of Willmore surfaces that arises when we consider extremals of the Willmore functional with respect to *infinitesimally conformal* variations,[1] rather than with respect to all variations. A variation $(\Lambda_t)_t$ of a surface Λ through null line subbundles of $\underline{\mathbb{R}}^{n+1,1}$ defining immersions of M into $\mathbb{P}(\mathcal{L})$ is said to be infinitesimally conformal if, fixing $Z \in \Gamma(T^{1,0}M)$ (respectively, $Z \in \Gamma(T^{0,1}M)$), locally never-zero, and, for each t, g_t in the conformal class of metrics induced in M by Λ_t, we have

$$\frac{d}{dt}_{|t=0} g_t(Z, Z) = 0,$$

[1] To which references as *conformal variations* can be found in the literature.

Conformal variations, characterized by the g_t-isotropy of $T^{1,0}M$ (respectively, $T^{0,1}M$), for all t, are, in particular, infinitesimally conformal variations. Constrained Willmore surfaces form a Möbius invariant class of surfaces with strong links to the theory of integrable systems, as we shall explore in this work.

F. Burstall et al. [**14**] established a manifestly conformally invariant characterization of constrained Willmore surfaces in space-forms, which, in particular, extended the concept of constrained Willmore to surfaces that are not necessarily compact. Chapter 5 is dedicated to deriving from the variational problem the reformulation of this characterization, by F. Burstall and D. Calderbank [**11**], presented below. The argument consists of a generalization to n-space of the argument presented in [**7**] for the particular case of $n = 3$. Set

$$\Lambda^{1,0} := \Lambda \oplus d\sigma(T^{1,0}M) \quad \text{and} \quad \Lambda^{0,1} := \Lambda \oplus d\sigma(T^{0,1}M),$$

independently of $\sigma \in \Gamma(\Lambda)$ never-zero, and then

$$\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}.$$

Cf. [**11**], Λ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with

$$d^{\mathcal{D}} q = 0, \tag{0.4}$$

such that

$$d^{\mathcal{D}} * \mathcal{N} = 2\,[q \wedge *\mathcal{N}]. \tag{0.5}$$

In this case, we may refer to Λ as, specifically, a q-constrained Willmore surface and to q as a [Lagrange] multiplier to Λ. Willmore surfaces are the 0-constrained Willmore surfaces. The zero multiplier is not necessarily the only multiplier to a constrained Willmore surface with no constraint on the conformal structure. In fact, in Chapter 8, we characterize isothermic constrained Willmore surfaces in space-forms by the non-uniqueness of multiplier. On the other hand, a classical result by Thomsen [**55**] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some space-form.

A multiplier to a surface Λ in the projectivized light-cone is, in particular, a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. For such a q, equations (0.4) and (0.5), together, encode the flatness of the metric connection

$$d_q^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2} - 1)\,q^{1,0} + (\lambda^2 - 1)\,q^{0,1},$$

on $(\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$, or, equivalently, for all $\lambda \in S^1$. Constrained Willmore surfaces in space-forms, admitting q as a multiplier, are characterized by the flatness of the S^1-family of metric connections d_q^λ on $\underline{\mathbb{R}}^{n+1,1}$, in an integrable systems interpretation due to F. Burstall and D. Calderbank [**11**]. This characterization will enable

us, in Chapter 6, to define a spectral deformation of constrained Willmore surfaces in space-forms, by the action of the loop of flat metric connections d^λ_q, as well as a *Bäcklund transformation*, by applying a dressing action.

Our transformations of constrained Willmore surfaces will be based on the *constrained harmonicity* of the central sphere congruence. Given $\hat{d}$ a flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and V a non-degenerate subbundle of $\underline{\mathbb{C}}^{n+2}$, we generalize naturally the decomposition (0.2) to a decomposition

$$\hat{d} = \hat{\mathcal{D}}_V + \hat{\mathcal{N}}_V$$

and, given $q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$, define then, for each $\lambda \in \mathbb{C}\backslash\{0\}$, a connection

$$\hat{d}^{\lambda,q}_V := \hat{\mathcal{D}}_V + \lambda^{-1}\hat{\mathcal{N}}^{1,0}_V + \lambda\hat{\mathcal{N}}^{0,1}_V + (\lambda^{-2}-1)q^{1,0} + (\lambda^2-1)q^{0,1},$$

on $\underline{\mathbb{C}}^{n+2}$, generalizing $d^\lambda_q = d^{\lambda,q}_S$. We define the bundle V to be $(q,\hat{d})$-*constrained harmonic* if $\hat{d}^{\lambda,q}_V$ is flat, for all $\lambda \in \mathbb{C}\backslash\{0\}$, or, equivalently, for all $\lambda \in S^1$. A simple, yet crucial, observation is that, given $\tilde{d}$ another flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and $\phi : (\underline{\mathbb{C}}^{n+2},\tilde{d}) \to (\underline{\mathbb{C}}^{n+2},\hat{d})$ an isomorphism of bundles provided with a metric and a connection, V is $(q,\tilde{d})$-constrained harmonic if and only if ϕV is $(\mathrm{Ad}_\phi q,\hat{d})$-constrained harmonic. The constrained harmonicity of a bundle applies to the central sphere congruence, providing a characterization of constrained Willmore surfaces in space-forms.

The transformations of a constrained Willmore surface Λ in the projectivized light-cone we present are, in particular, pairs $((\Lambda^{1,0})^*,(\Lambda^{0,1})^*)$ of transformations $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ of $\Lambda^{1,0}$ and $\Lambda^{0,1}$, respectively. The fact that $\Lambda^{1,0}$ and $\Lambda^{0,1}$ intersect in a rank 1 bundle will ensure that $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ have the same property. The isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ will ensure that of $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ and, therefore, of their intersection. The reality of the bundle $\Lambda^{1,0}\cap\Lambda^{0,1}$ and the fact that it defines an immersion of M into $\mathbb{P}(\mathcal{L})$ are preserved by the spectral deformation, but it is not clear that the same is necessarily true for Bäcklund transformations. This motivates us to define *complexified surface* and, thereafter, *complexified constrained Willmore surface*.

The spectral deformation defined by the action of the loop of flat metric connections d^λ_q coincides, up to reparametrization, with the one presented in [**14**]. More interestingly, we define a Bäcklund transformation of constrained Willmore surfaces in space-forms. We use a version of the dressing action theory of C.-L. Terng and K. Uhlenbeck [**54**]. We start by defining a local action of a group of rational maps on the set of flat metric connections of the type $\hat{d}^{\lambda,q}_S$, with $\hat{d}$ flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and $q \in \Omega^1(\wedge^2 S \oplus \wedge^2 S^\perp)$. Namely, given $r = r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ holomorphic at $\lambda = 0$ and $\lambda = \infty$ and twisted in the sense that $\rho r(\lambda)\rho = r(-\lambda)$, for ρ reflection across S, we define a 1-form $\hat{q}$ with values in $\wedge^2 S$ (note that the fact that $r(\lambda)$ is twisted establishes that both $r(0)$ and $r(\infty)$ preserve S) by

$$\hat{q}^{1,0} := \mathrm{Ad}_{r(0)}q^{1,0}, \quad \hat{q}^{\,0,1} := \mathrm{Ad}_{r(\infty)}q^{0,1},$$

and a new family of metric connections from $d_S^{\lambda,q}$ by $\hat{d}_S^{\lambda,\hat{q}} := r(\lambda) \circ d_S^{\lambda,q} \circ r(\lambda)^{-1}$. Obviously, for each λ, the flatness of $\hat{d}_S^{\lambda,\hat{q}}$ is equivalent to that of $d_S^{\lambda,q}$. Crucially, if $\hat{d}_S^{\lambda,\hat{q}}$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$, then the notation $\hat{d}_S^{\lambda,\hat{q}}$ proves to be not merely formal, for $\hat{d} := \hat{d}_S^{1,\hat{q}}$. In that case, it follows that, if Λ is q-constrained Willmore, then S is $(\hat{q},\hat{d})$-constrained harmonic and, therefore, in the case $1 \in \mathrm{dom}(r)$, $S^* := r(1)^{-1}S$ is q^*-constrained harmonic, for

$$q^* := \mathrm{Ad}_{r(1)^{-1}}\hat{q}.$$

The transformation of S into S^*, preserving constrained harmonicity, leads, furthermore, to a transformation of Λ into a new constrained Willmore surface, provided that

$$\det r(0)|_S = \det r(\infty)|_S. \tag{0.6}$$

Set

$$(\Lambda^*)^{1,0} := r(1)^{-1}r(\infty)\Lambda^{1,0}, \quad (\Lambda^*)^{0,1} := r(1)^{-1}r(0)\Lambda^{0,1}$$

and

$$\Lambda^* := (\Lambda^*)^{1,0} \cap (\Lambda^*)^{0,1}.$$

Condition (0.6) establishes Λ^* as a line bundle (the argument is based on the two families of lines on the Klein quadric). The isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ ensures that of Λ^*. It is not clear, though, that Λ^* is a real bundle. If Λ^* is a *real surface*, one proves that S^* is the central sphere congruence of Λ^* and that the bundles $(\Lambda^*)^{1,0}$ and $(\Lambda^*)^{0,1}$ defined above are not merely formal. The fact that q is a multiplier to Λ establishes, furthermore, $q^{1,0} \in \Omega^{1,0}(\wedge^2\Lambda^{0,1})$ and, therefore, $(q^*)^{1,0} \in \Omega^{1,0}(\wedge^2(\Lambda^*)^{0,1}) \subset \Omega^{1,0}(\Lambda^* \wedge (\Lambda^*)^{(1)})$. We conclude that, if, furthermore, q^* is real, then Λ^* is a q^*-constrained Willmore surface.

We then construct rational maps $r(\lambda)$ satisfying the hypothesis of the dressing action, together with reality preserving conditions. As the philosophy underlying the work of C.-L. Terng and K. Uhlenbeck [**54**] suggests, we consider linear fractional transformations. We define two different types of such transformations, *type* p and *type* q, each one of them satisfying the hypothesis of the dressing action with the exception of condition (0.6). Iterating the procedure, in a 2-step process composing the two different types of transformations, will produce a desired $r(\lambda)$. A *Bianchi permutability* of type p and type q transformations of constrained harmonic bundles is established. For special choices of parameters, the reality of Λ as a bundle proves to establish that of Λ^*, whilst the reality of q establishes that of q^*. For such a choice of parameters, Λ^* is said to be a *Bäcklund transform* of Λ, provided that it immerses. For future reference, it is useful to know that Bäcklund transformation parameters are pairs α, L^α with, in particular, $\alpha \in \mathbb{C}$ and $L^\alpha \subset \underline{\mathbb{C}}^{n+2}$, and that parameters α, L^α and $-\alpha, \rho L^\alpha$ give rise to the same transform. Note that both Bäcklund transformation and

spectral deformation corresponding to the zero multiplier preserve the class of Willmore surfaces.

In Chapter 6, we define, more generally, a spectral deformation and a Bäcklund transformation of constrained harmonic bundles and of complexified constrained Willmore surfaces. We complete the chapter by establishing a permutability between spectral deformation and Bäcklund transformation.

In Chapter 7, we introduce the concept of *conserved quantity* of a constrained Willmore surface in a space-form, an idea by F. Burstall and D. Calderbank. A conserved quantity of Λ consists of a Laurent polynomial

$$p(\lambda) = \lambda^{-1} v + v_0 + \lambda \overline{v}$$

with $v_0 \in \Gamma(S)$ real, $v \in \Gamma(S^\perp)$ and $p(1) = v_0 + v + \overline{v} \in \Gamma(\underline{\mathbb{R}}^{n+1,1})$ non-zero, which is $d_S^{\lambda,q}$-parallel, for all $\lambda \in \mathbb{C}\backslash\{0\}$ and some multiplier q to Λ. Constrained Willmore surfaces in space-forms admitting a conserved quantity form a subclass of constrained Willmore surfaces preserved by both spectral deformation and Bäcklund transformation, for special choices of parameters.

The existence of a conserved quantity $p(\lambda)$ of Λ establishes, in particular, the constancy of $p(1)$. In the particular case of $n = 3$, we verify that Λ has constant mean curvature in the space-form $S_{p(1)}$, that is, the surface defined by Λ in the space-form $S_{p(1)}$ has constant mean curvature. In fact, in codimension 1, the class of constrained Willmore surfaces in space-forms admitting a conserved quantity consists of the class of constant mean curvature surfaces in space-forms. Another example of constrained Willmore surfaces admitting a conserved quantity is that of surfaces with holomorphic mean curvature vector in some space-form in codimension 2. In codimension 2, the complexification of $S^\perp$ admits a unique decomposition into the direct sum of two null complex lines, complex conjugate of each other: given $v \in \Gamma(S^\perp)$ null, $S^\perp = \langle v \rangle \oplus \langle \overline{v} \rangle$. Such a v defines an almost-complex structure

$$J_v := I \begin{cases} i & \text{on } \langle v \rangle \\ -i & \text{on } \langle \overline{v} \rangle \end{cases}$$

on $S^\perp$. F. Burstall and D. Calderbank [**11**] proved that a codimension 2 surface in a space-form, with holomorphic mean curvature vector with respect to the complex structure induced by $\nabla^{S^\perp}$, is constrained Willmore. We prove it in our setting, proving, furthermore, that, in 4-dimensional space-form, the constrained Willmore surfaces admitting a conserved quantity $p(\lambda) = \lambda^{-1} v + v_0 + \lambda \overline{v}$ with v null are the surfaces with holomorphic mean curvature vector in the space-form $S_{p(1)}$, with respect to the complex structure on $(S^\perp, J_v, \nabla^{S^\perp})$ determined by Koszul-Malgrange Theorem.

Chapter 8 is dedicated to relating the class of constrained Willmore surfaces to the isothermic condition. The class of isothermic surfaces is a Möbius invariant class of

surfaces. A manifestly conformally invariant formulation of the isothermic condition, by F. Burstall and U. Pinkall [**13**], characterizes isothermic surfaces by the existence of a non-zero real closed 1-form η with values in a certain subbundle of the skew-symmetric endomorphisms of $\underline{\mathbb{R}}^{n+1,1}$. We establish the set of multipliers to an isothermic q-constrained Willmore surface (Λ, η) as the 1-dimensional affine space $q + \langle *\eta \rangle_{\mathbb{R}}$. The constrained Willmore spectral deformation is known to preserve the isothermic condition, cf. [**14**]. We derive it in our setting. As for Bäcklund transformation of constrained Willmore surfaces, we believe it does not necessarily preserve the isothermic condition. This shall be the subject of further work.

Following the work of F. Burstall, D. Calderbank and U. Pinkall [**11**], [**13**], we characterize isothermic surfaces by the flatness of a certain $\mathbb{R}$-family of metric connections on $\underline{\mathbb{R}}^{n+1,1}$ and define, in terms of this family of connections, both the isothermic spectral deformation, discovered in the classical setting by Calapso and, independently, by Bianchi; and the isothermic Darboux transformation.

We dedicate a section to the special class of constant mean curvature (CMC) surfaces in 3-dimensional space-forms. CMC surfaces in 3-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by J. Richter [**51**], with constrained Willmore Bäcklund transformations; both constrained Willmore and isothermic spectral deformations; as well as a spectral deformation of their own and, in the Euclidean case, isothermic Darboux transformations and Bianchi-Bäcklund transformations. The isothermic spectral deformation is known to preserve the constancy of the mean curvature of a surface in some space-form, cf. [**14**]. Characterized as the class of constrained Willmore surfaces in 3-dimensional space-forms admitting a conserved quantity, the class of CMC surfaces in 3-space is known to be preserved by both constrained Willmore spectral deformation and Bäcklund transformation, for special choices of parameters. We verify that both the space-form and the mean curvature are preserved by constrained Willmore Bäcklund transformation and investigate how these change under constrained Willmore and isothermic spectral deformation. We present the classical CMC spectral deformation by means of the action of a loop of flat metric connections on the class of CMC surfaces in 3-space (preserving the space-form and the mean curvature) and observe that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation. These spectral deformations of CMC surfaces in 3-space are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation. S. Kobayashi and J.-I. Inoguchi [**35**] proved that isothermic Darboux transformation of CMC surfaces in Euclidean 3-space is equivalent to Bianchi-Bäcklund transformation. We believe isothermic Darboux transformation of a CMC surface in Euclidean 3-space can be obtained as a particular case of constrained Willmore Bäcklund transformation. This shall be the subject of further work.

We present a 1-form η_∞, derived by F. Burstall and D. Calderbank from a surface Λ with constant mean curvature H_∞ in a space-form S_{v_∞}, which establishes (Λ, η_∞) as an isothermic surface and for which scaling by H_∞ provides a multiplier,

$$q_\infty := H_\infty \eta_\infty,$$

to Λ; and, for each $t \in \mathbb{R}$ and

$$q_\infty^t := q_\infty + t * \eta_\infty,$$

we establish a q_∞^t-conserved quantity to Λ.

Lastly, we dedicate Chapter 9 to the special case of surfaces in 4-space. Our approach is quaternionic, based on the model of the conformal 4-sphere on the quaternionic projective space, and follows the work of F. Burstall et al. [**12**]. We consider the natural identification of $\mathbb{H}$ with $\mathbb{R}^4$ and then the natural identification of $\mathbb{H}^2$ with $\langle 1, i\rangle^4 = \mathbb{C}^4$. We provide $\wedge^2\mathbb{C}^4$ with the real structure $\wedge^2 j$ and define a metric on $\wedge^2\mathbb{C}^4$ by $(v_1 \wedge v_2, v_3 \wedge v_4) := -\det(v_1, v_2, v_3, v_4)$, for $v_1, v_2, v_3, v_4 \in \mathbb{C}^4$, with $\det(v_1, v_2, v_3, v_4)$ denoting the determinant of the matrix whose columns are the components of v_1, v_2, v_3 and v_4, respectively, on the canonical basis of $\mathbb{C}^4$. This metric induces a metric with signature $(5, 1)$ on the space of real vectors of $\wedge^2\mathbb{C}^4$,

$$\mathrm{Fix}(\wedge^2 j) = \mathbb{R}^{5,1}.$$

Via the Plücker embedding, we identify a j-stable 2-plane L in $\mathbb{C}^4$ with the real null line $\wedge^2 L$ in $(\mathrm{Fix}(\wedge^2 j))^{\mathbb{C}}$, presenting, in this way, the quaternionic projective space $\mathbb{H}P^1$ as a model for the conformal 4-sphere,

$$\mathbb{H}P^1 \cong S^4.$$

Surfaces in S^4 are described in this model as the immersed bundles

$$L \cong \wedge^2 L : M \to S^4$$

of j-stable 2-planes in $\mathbb{C}^4$. As we are in codimension 2, the complexification of $S^\perp$ admits a unique decomposition $S^\perp = S^\perp_+ \oplus S^\perp_-$ into the direct sum of two null complex lines, complex conjugate of each other. Via the Plücker embedding, we identify $S^\perp_+$ with some bundle S_+ of 2-planes in $\mathbb{C}^4$, and write then

$$S = S_+ \wedge jS_+.$$

We define a j-commuting complex structure on $\underline{\mathbb{C}}^4$, which we still denote by S, by the condition of admitting S_+ as the eigenspace associated to the eigenvalue i (and, therefore, jS_+ as the eigenspace associated to $-i$), together with a certain condition on the sign.

Following the characterization of the harmonicity of S presented in [**12**], we establish, more generally, a characterization of constrained Willmore surfaces in S^4 in

terms of the closeness of a certain form, as follows. Under the standard identification $sl(\mathbb{C}^4) \cong o(\wedge^2\mathbb{C}^4)$, 1-forms with values in $\Lambda \wedge \Lambda^{(1)}$ correspond to S-commuting 1-forms with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$. For such a form q, condition $d^{\mathcal{D}}q = 0$ establishes $Sq = *q$. A surface L in S^4 is a q-constrained Willmore surface, for some 1-form q with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$ such that $Sq = *\,q = qS$, if and only if

$$d * (Q + q) = 0,$$

for the Hopf field $Q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4))$. The closeness of the 1-form $*(Q+q)$ ensures the existence of $G \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ with $dG = 2 * (Q+q)$, as well as the integrability of the Riccati equation

$$dT = \rho T(dG)T - dF + 4\rho q T,$$

for each $\rho \in \mathbb{R}\backslash\{0\}$, fixing such a G and setting $F := G - S$. For a local solution $T \in \Gamma(Gl_j(\underline{\mathbb{C}}^4))$ of the ρ-Riccati equation, we define the *Darboux transform* of L of parameters ρ, T by setting

$$\hat{L} := T^{-1}L,$$

extending, in this way, the Darboux transformation of Willmore surfaces in S^4 presented in [**12**] to a transformation of constrained Willmore surfaces in 4-space.

We apply, yet again, the dressing action presented in Chapter 6 to define another transformation of constrained Willmore surfaces in 4-space, the *untwisted Bäcklund transformation*, referring then to the original one as the *twisted Bäcklund transformation*. We verify that, when both are defined, twisted and untwisted Bäcklund transformations coincide. We establish a correspondence between Darboux transformation parameters ρ, T with $\rho > 1$ and pairs α, L^α; $-\alpha, \rho_V L^\alpha$ of untwisted Bäcklund transformation parameters with α^2 real, and show that the corresponding transformations coincide. Darboux transformation of constrained Willmore surfaces with respect to parameters ρ, T with $\rho \leq 1$ is trivial. Non-trivial Darboux transformation of constrained Willmore surfaces in 4-space is, in this way, established as a particular case of constrained Wilmore Bäcklund transformation.

The main new or, at least, unpublished[2] (to my knowledge) notions and results presented in this work can be listed as follows:

- Section 6.1 (we define *constrained harmonicity* of a bundle, which will apply to the central sphere congruence to provide a characterization of constrained Willmore surfaces in space-forms);
- Sections 6.2 and 6.3 (we define *complexified surface*, in generalization of surface in a space-form, defining, thereafter, *complexified constrained Willmore surface*);

[2]With the trivial exception of the PhD Thesis on which this work is based.

- Section 6.4 (we define a $\mathbb{C}\backslash\{0\}$-spectral deformation of complexified constrained Willmore surfaces by the action of a family of flat metric connections; for unit parameter, this deformation preserves reality conditions and, when restricted to *real* surfaces, it coincides, up to reparametrization, with the one presented in [**14**]);
- Sections 6.5 and 6.6 (we define a *Bäcklund transformation* of complexified constrained Willmore surfaces, by applying a dressing action; for special choices of parameters, this transformation preserves reality conditions);
- Section 6.7 (we establish a permutability between Bäcklund transformation and spectral deformation of complexified constrained Willmore surfaces);
- Proposition 6.36 (we prove that the quadratic differential is preserved under the corresponding Bäcklund transformation);
- Section 7.1 (we introduce the concept of *conserved quantity* of a constrained Willmore surface in a space-form, an idea by F. Burstall and D. Calderbank);
- Theorems 7.6 and 7.7 (we prove that the class of constrained Willmore surfaces in 3-dimensional space-forms admitting a conserved quantity is preserved by both spectral deformation and Bäcklund transformation, for special choices of parameters);
- Section 7.2.2 (we prove that, in codimension 2, surfaces with holomorphic mean curvature vector in some space-form (already known to be constrained Willmore, cf. [**11**]) are examples of constrained Willmore surfaces admitting a conserved quantity);
- Proposition 8.20 and Theorem 8.22 (we establish the class of CMC surfaces in 3-dimensional space-forms as the class of constrained Willmore surfaces in 3-space admitting a conserved quantity);
- Sections 8.2.3 and 8.2.4 (we verify that both the space-form and the mean curvature are preserved by constrained Willmore Bäcklund transformation of CMC surfaces in 3-dimensional space-forms and investigate how these change under constrained Willmore spectral deformation; we verify that the classical CMC spectral deformation of a CMC surface in 3-space can be obtained as composition of isothermic and constrained Willmore spectral deformation);
- Section 8.1.4 (we characterize isothermic constrained Willmore surfaces in space-forms by the non-uniqueness of multiplier and establish the set of multipliers to an isothermic q-constrained Willmore surface (Λ,η) as the affine space $q+\langle *\eta\rangle_{\mathbb{R}}$);
- Theorems 9.12 and 9.13 (we provide two (equivalent) characterizations of constrained Willmore surfaces in 4-space in the quaternionic setting);

- Section 9.3.3 (we extend the Darboux transformation of Willmore surfaces in S^4 presented in [**12**] to a transformation of constrained Willmore surfaces in 4-space);
- Section 9.3.4 (we prove that Darboux transformation of parameters ρ, T with $\rho > 1$ is equivalent to Bäcklund transformation of parameters α, L^α with α^2 real; Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial).

Our theory is local and, throughout the text, with no need for further reference, restriction to a suitable non-empty open set shall be underlying.

Foundations of Differential Geometry, Vol.s 1,2, by S. Kobayashi and K. Nomizu [**36**], *Riemannian Geometry*, by T. Willmore [**59**], and *Selected topics in Harmonic maps*, by J. Eells and L. Lemaire [**23**], are good references to the basic background.

CHAPTER 1

A bundle approach to conformal surfaces in space-forms

Our study is a study of geometrical aspects that are invariant under Möbius transformations.[1] We present a Möbius description of space-forms in the projectivized light-cone, following [**10**]. Such description is based on the model of the conformal n-sphere on the projective space $\mathbb{P}(\mathcal{L})$ of the light-cone $\mathcal{L}$ of $\mathbb{R}^{n+1,1}$, due to Darboux [**21**], which, in particular, yields a conformal description of Euclidean n-spaces and hyperbolic n-spaces as submanifolds of $\mathbb{P}(\mathcal{L})$. With this, we approach a surface conformally immersed in n-space as a null line subbundle Λ of $\underline{\mathbb{R}}^{n+1,1} = M \times \mathbb{R}^{n+1,1}$ defining an immersion $\Lambda : M \to \mathbb{P}(\mathcal{L})$ of an oriented surface M, which we provide with the conformal structure induced by Λ, into the projectivized light-cone. The realization of all space-forms as submanifolds of the projectivized light-cone considered in this text arises from the realization, cf. [**10**], of all n-dimensional space-forms as conic sections $S_{v_\infty} := \{v \in \mathcal{L} : (v, v_\infty) = -1\}$ of the light-cone, with $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero. S_{v_∞} inherits a positive definite metric of constant sectional curvature $-(v_\infty, v_\infty)$ from $\mathbb{R}^{n+1,1}$ and is either a copy of a sphere, a copy of Euclidean space or two copies of hyperbolic space, according to the sign of (v_∞, v_∞). For each v_∞, the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defines a diffeomorphism $\pi|_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L} \cap \langle v_\infty\rangle^\perp)$. We provide $\mathbb{P}(\mathcal{L})$ with the conformal structure of the (positive definite) metric induced by $\pi|_{S_{v_\infty}}$, fixing v_∞ time-like, independently of the choice of v_∞, and, in this way, identify $\mathbb{P}(\mathcal{L})$ with the conformal n-sphere and make each diffeomorphism $\pi|_{S_{v_\infty}}$ - for a general v_∞, not necessarily time-like - into a conformal diffeomorphism.

By *metric* on a $\mathbb{K}$-vector bundle P, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we mean a non-degenerate section of $S^2(P, \mathbb{K})$. With no need for further reference, given a vector bundle P, provided with a metric, the metric considered on the complexification $P^{\mathbb{C}}$ of P shall be the complex bilinear extension of the metric on P. In particular,

$$\overline{(\alpha_1 + i\beta_1, \alpha_2 + i\beta_2)} = (\,\overline{\alpha_1 + i\beta_1}, \overline{\alpha_2 + i\beta_2}\,),$$

for $\alpha_i, \beta_i \in \Gamma(P)$, for $i = 1, 2$. More generally, throughout this text, when appropriated and unless indication to the contrary is provided, we will move from real tensors to complex tensors by complex multilinear extension, preserving, therefore, reality conditions and, in general, notation. With no need for further reference, given V and W

[1]With the exception of the study of constant mean curvature surfaces, in Sections 7.2.1 and 8.2 below.

vector bundles provided with a metric, we shall consider $\mathrm{Hom}(V,W)$ provided with the canonical metric induced by V and W, the metric given by

$$(\alpha,\beta) := \mathrm{tr}\,(\beta^t\alpha),$$

for $\alpha,\beta\in\Gamma(\mathrm{Hom}(V,W))$, with β^t denoting the transpose of β.

As usual, by metric on a manifold M we mean a metric on the tangent bundle TM, as a $\mathbb{R}$-bundle. Two positive definite metrics g and g' on a manifold M are said to be *conformally equivalent* if $g' = e^u g$, for some $u\in C^\infty(M;\mathbb{R})$. A class of conformally equivalent Riemannian metrics on M is said to be a *conformal structure* on M. When provided with a conformal structure, M is said to be a *conformal manifold*. A mapping ϕ of M into a manifold N provided with a metric $(\,,)_N$ defines a section g_ϕ of $S^2(TM,\mathbb{R})$, given by

$$g_\phi(X,Y) := (d_X\phi, d_Y\phi)_N,$$

for $X,Y\in\Gamma(TM)$, which we refer to as the *metric induced in M by ϕ*. If the metric $(\,,)_N$ is positive definite and ϕ is an immersion, then the metric g_ϕ is positive definite. In the case M is provided with a Riemannian metric g, $\phi:(M,g)\to(N,(\,,)_N)$ is said to be *conformal* if g_ϕ is a positive definite metric conformally equivalent to g.

Our study is a study of geometrical aspects that are invariant under conformal diffeomorphisms or *Möbius transformations*. It is, in particular, a study of angle-preserving transformations, dedicated to submanifolds of an ambient space equipped with a conformal class of metrics but not carrying a distinguished metric. Our point of view is that of *Möbius geometry*, where there is an angle measurement, but, in contrast to Euclidean geometry, there is no measurement of distances. This lack of length measurement has interesting consequences. For example, from the point of view of Möbius geometry, $S^n\backslash\{x\}$ and $\mathbb{R}^n$ are no longer distinguished, as the stereographic projection of pole $x\in S^1$, of $S^n\backslash\{x\}$ onto $\mathbb{R}^n$, is a conformal diffeomorphism. We refer to invariance under Möbius transformations as *Möbius invariance* or *conformal invariance*.

For $m\in\mathbb{N}$, let $\mathbb{R}^{m,1}$ be the $(m+1)$-dimensional Minkowski space with signature $(m,1)$, i.e., a real $(m+1)$-dimensional vector space equipped with a metric $(\,,\,)$ with respect to which there exists an orthogonal basis $e_1,...,e_{m+1}$ with

$$(e_i,e_i) = \left\{\begin{array}{ll} 1 & i<m+1 \\ -1 & i=m+1 \end{array}\right. .$$

As usual, we refer to a vector $v\in\mathbb{R}^{m,1}$ such that $(v,v)=0$ (respectively, $(v,v)<0$, $(v,v)>0$) as a light-like (respectively, time-like, space-like) vector. Let $\mathcal{L}$ be the light-cone in $\mathbb{R}^{m,1}$,

$$\mathcal{L} = \{v\in\mathbb{R}^{m,1}\backslash\{0\} : (v,v)=0\},$$

an m-dimensional submanifold of $\mathbb{R}^{m,1}$, and, in the case $m > 1$, let $\mathcal{L}^+$ and $\mathcal{L}^-$ be the connected components of $\mathcal{L}$.[2] Let $\mathbb{P}(\mathcal{L})$ be the projectivized light-cone,

$$\mathbb{P}(\mathcal{L}) = \{\langle v \rangle : v \in \mathcal{L}\},$$

an $(m-1)$-dimensional submanifold of $\mathbb{P}(\mathbb{R}^{m,1})$.

Recall the sectional curvature $K(\wp)$ of a 2-plane $\wp$ in the tangent space T_xM to a Riemannian manifold M at a point $x \in M$,

$$K(\wp) = -(R(X_1, X_2)X_1, X_2),$$

for R the curvature tensor of M (provided with the Levi-Civita connection), independently of the choice of an orthonormal basis X_1, X_2 of T_xM. It is opportune to establish some (usual) notation: given X, Y, Z, W in $\Gamma(TM)$,

$$R(X, Y, Z, W) := -(R(X, Y)Z, W).$$

If $K(\wp)$ is constant for all planes $\wp$ in T_xM and for all points $x \in M$, then M is said to be a space of constant curvature. If $K(\wp)$ is constant for all planes $\wp$ in T_xM but possibly depends on $x \in M$, we say that M has constant sectional curvature $K = K(x)$, with $x \in M$. That is always the case when M is 2-dimensional, in which case K is famously said to be the Gaussian curvature of the surface M.

By *space-form* we mean a connected and simply connected complete Riemannian manifold of constant sectional curvature. For simplicity, we may use *n-space* to refer to n-dimensional space-form. Two space-forms with the same curvature are isometric to each other. Fix $n \in \mathbb{N}$. Our model of flat n-space is the Euclidean space $\mathbb{R}^n$. Given $r \in \mathbb{R}^+$, our model of n-space of curvature $1/r^2$ is the n-sphere

$$S^n(r) := \{x \in \mathbb{R}^{n+1} : (x, x) = r^2\},$$

whereas that of n-space of curvature $-1/r^2$ is the hyperbolic n-space consisting of either of the two connected components of

$$H^n(r) := \{x \in \mathbb{R}^{n,1} : (x, x) = -r^2\}.$$

1.1. Space-forms in the conformal projectivized light-cone

Following [**10**], we present a Möbius description of spheres, Euclidean spaces and hyperbolic spaces in the projectivized light-cone. We start by realizing all n-dimensional space-forms as connected components of conic sections of $\mathcal{L} \subset \mathbb{R}^{n+1,1}$.

Let $\mathcal{L}$ be the light-cone in $\mathbb{R}^{n+1,1}$. Fix $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero. Set

$$S_{v_\infty} := \{v \in \mathcal{L} : (v, v_\infty) = -1\},$$

[2]Non-collinear elements $v_0, v_\infty \in \mathcal{L}$ are in the same component if and only if $(v_0, v_\infty) < 0$.

the conic section of the light-cone given by the intersection of $\mathcal{L}$ with the hyperplane $\{v \in \mathbb{R}^{n+1,1} : (v, v_\infty) = -1\}$ of $\mathbb{R}^{n+1,1}$, which is an n-dimensional submanifold of $\mathbb{R}^{n+1,1}$. Given $v \in \mathcal{L}$,

$$T_v\mathcal{L} = \langle v \rangle^\perp$$

and, for $v \in S_{v_\infty}$,

$$T_v S_{v_\infty} = \langle v, v_\infty \rangle^\perp.$$

The fact that $(v, v_\infty) \neq 0$ establishes the non-degeneracy of the subspace $\langle v, v_\infty \rangle$ of $\mathbb{R}^{n+1,1}$, or, equivalently, a decomposition

$$\mathbb{R}^{n+1,1} = \langle v, v_\infty \rangle \oplus T_v S_{v_\infty}. \tag{1.1}$$

The nullity of v establishes then $\langle v, v_\infty \rangle$ as a 2-dimensional space with a metric with signature $(1, 1)$, establishing, therefore, $T_v S_{v_\infty}$ as isometric to $\mathbb{R}^n$. S_{v_∞} inherits a positive definite metric from $\mathbb{R}^{n+1,1}$.

Proposition 1.1. *S_{v_∞} is either a copy of an n-sphere, a copy of Euclidean n-space or two copies of hyperbolic n-space, according to the sign of (v_∞, v_∞).*

In the proof of the proposition, we will show, in particular, that S_{v_∞} has constant sectional curvature $-(v_\infty, v_\infty)$, and that, if v_∞ is space-like, then $S_{v_\infty} \cap \mathcal{L}^+$ is a copy of hyperbolic n-space, as well as $S_{v_\infty} \cap \mathcal{L}^-$.

Proof. If v_∞ is light-like, then, choosing $v_0 \in S_{v_\infty}$, the map $v \mapsto v - v_0 + (v, v_0)v_\infty$ defines an isometry

$$S_{v_\infty} \to \langle v_0, v_\infty \rangle^\perp \cong \mathbb{R}^n$$

whose inverse is given by $x \to x + v_0 + \frac{1}{2}(x, x)v_\infty$.

Now contemplate the case v_∞ is not light-like. In that case, $\langle v_\infty \rangle$ is non-degenerate, so we have a decomposition $\mathbb{R}^{n+1,1} = \langle v_\infty \rangle \oplus \langle v_\infty \rangle^\perp$. Set $r' := (v_\infty, v_\infty)^{-1}$. For $v \in S_{v_\infty}$, write $v = -r'v_\infty + v^\perp$ with $v^\perp \perp v_\infty$ and note that $0 = (v, v) = r' + (v^\perp, v^\perp)$. In the particular case v_∞ is time-like, the projection $v \mapsto v^\perp$ defines a diffeomorphism

$$S_{v_\infty} \to S^n(r) \subset \langle v_\infty \rangle^\perp \cong \mathbb{R}^{n+1}$$

onto the sphere of radius $r := \sqrt{-r'}$ in $\langle v_\infty \rangle^\perp \cong \mathbb{R}^{n+1}$, which we easily check to be an isometry. In the case v_∞ is space-like, the projection $v \mapsto v^\perp$ defines an isometry

$$S_{v_\infty} \to H^n(r) \subset \langle v_\infty \rangle^\perp \cong \mathbb{R}^{n,1},$$

for $r := \sqrt{r'}$. $\square$

Now contemplate the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$, which we may, alternatively, denote by $\pi_{\mathcal{L}}$. Note that, given $v \in \mathcal{L}$,

$$\operatorname{Ker} d\pi_v = \langle v \rangle, \tag{1.2}$$

so we have an isomorphism $d\pi_v : \langle v\rangle^\perp/\langle v\rangle \cong d\pi_v(T_v\mathcal{L})$, given by $u + \langle v\rangle \mapsto d\pi_v(u)$, for $u \in \langle v\rangle^\perp$. As $d\pi_v(T_v\mathcal{L})$ is an n-dimensional subspace of $T_{\langle v\rangle}\mathbb{P}(\mathcal{L})$, we conclude that $T_{\langle v\rangle}\mathbb{P}(\mathcal{L}) = d\pi_v(T_v\mathcal{L})$,

$$d\pi_v : \langle v\rangle^\perp/\langle v\rangle \cong T_{\langle v\rangle}\mathbb{P}(\mathcal{L}) \tag{1.3}$$

and, therefore, that π is a submersion.

The map π defines a diffeomorphism

$$\pi|_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L}\cap\langle v_\infty\rangle^\perp) \tag{1.4}$$

whose inverse is given by

$$\langle v\rangle \mapsto S_{v_\infty} \cap \langle v\rangle = \frac{-1}{(v, v_\infty)}\, v.$$

Lemma 1.2. *If v_∞ is time-like, then $\mathcal{L}\cap\langle v_\infty\rangle^\perp = \emptyset$. If v_∞ is light-like, then $\mathcal{L}\cap\langle v_\infty\rangle^\perp = \langle v_\infty\rangle$.*

Proof. If v_∞ is time-like, then $\langle v_\infty\rangle^\perp$ is space-like and, therefore, contains no light-like vectors.

On the other hand, recalling that there are no null subspaces of $\mathbb{R}^{n+1,1}$ with dimension higher than 1, we conclude that, if v_∞ is light-like, then $(v, v_\infty) \neq 0$, for all v in $\mathcal{L}\backslash\langle v_\infty\rangle$, completing the proof. $\square$

Hence:

Proposition 1.3. *Once restricted to S_{v_∞}, the canonical projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ defines a diffeomorphism onto $\mathbb{P}(\mathcal{L})$, $\mathbb{P}(\mathcal{L})\backslash\{\langle v_\infty\rangle\}$ or $\mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L}\cap\langle v_\infty\rangle^\perp)$, according as v_∞ is time-like, light-like or space-like, respectively.*

This will lead us to a conformal description of spheres, Euclidean spaces and hyperbolic spaces in the projectivized light-cone. Our next step is to provide the projectivized light-cone with a conformal structure.

We provide $\mathbb{P}(\mathcal{L})$ with the conformal structure $\mathcal{C}_{\mathbb{P}(\mathcal{L})}$ defined by the metric g_σ, fixing σ a never-zero section of the tautological bundle $\mathbb{P}(\mathcal{L})_{\mathbb{P}(\mathcal{L})} = (l)_{l\in\mathbb{P}(\mathcal{L})}$. This is well-defined and, furthermore, makes each diffeomorphism (1.4) into a conformal diffeomorphism. Indeed, given $\sigma, \sigma' : \mathbb{P}(\mathcal{L}) \to \mathbb{R}^{n+1,1}$ never-zero sections of $\mathbb{P}(\mathcal{L})_{\mathbb{P}(\mathcal{L})}$, we have $\sigma' = f\sigma$, for some never-zero $f \in C^\infty(\mathbb{P}(\mathcal{L})_{\mathbb{P}(\mathcal{L})}, \mathbb{R})$, and then

$$g_{\sigma'}(X, Y) = (f d_X\sigma + (d_X f)\sigma, f d_Y\sigma + (d_Y f)\sigma) = f^2 g_\sigma(X, Y),$$

as (σ, σ) vanishes and, therefore, so does $(\sigma, d\sigma)$. On the other hand, for non-zero $v_\infty \in \mathbb{R}^{n+1,1}$, the diffeomorphism $(\pi|_{S_{v_\infty}})^{-1} : \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L}\cap\langle v_\infty\rangle^\perp) \to S_{v_\infty}$ is a never-zero section of $\mathbb{P}(\mathcal{L})_{\mathbb{P}(\mathcal{L})}$ inducing in $\mathbb{P}(\mathcal{L})$ a positive definite metric from the one in S_{v_∞}.

We refer to the unit sphere S^n in $\mathbb{R}^{n+1}$, when provided by the conformal class of the round metric, as the conformal n-sphere. By providing $\mathbb{P}(\mathcal{L})$ with the conformal structure $\mathcal{C}_{\mathbb{P}(\mathcal{L})}$, we make the map $\pi|_{S_{v_\infty}} : S^n \cong S_{v_\infty} \to \mathbb{P}(\mathcal{L})$, defined by

$$x \mapsto \langle v_\infty + x \rangle,$$

for v_∞ unit time-like, into a conformal diffeomorphism, identifying the conformal n-sphere with the conformal projectivized light-cone,

$$S^n \cong \mathbb{P}(\mathcal{L}),$$

in a model due to Darboux [**21**]. In this model of the conformal n-sphere, the k-spheres S^k in S^n (intersections of S^n with $(k+1)$-dimensional affine spaces), for $0 \leq k \leq n$, are described as the submanifolds $\mathbb{P}(\mathcal{L} \cap V)$ with V non-degenerate $(k+2)$-dimensional subspace of $\mathbb{R}^{n+1,1}$ with signature $(k+1,1)$ (see, for example, [**53**], Appendix A); hyperbolic n-spaces are described as the submanifolds consisting of either of the two connected components of

$$S^n \backslash S^{n-1} \cong \mathbb{P}(\mathcal{L}) \backslash \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$$

with v_∞ space-like; and Euclidean n-spaces are described as those consisting of

$$S^n \backslash \{x\} \cong \mathbb{P}(\mathcal{L}) \backslash \{\langle v_\infty \rangle\}$$

with $x \in S^n$ and v_∞ light-like.

1.2. Conformal surfaces in space-forms

Throughout this text, let $n \geq 3$, $\mathcal{L}$ be the light-cone of $\mathbb{R}^{n+1,1}$, M be an oriented surface and $\underline{\mathbb{R}}^{n+1,1}$ denote the trivial bundle $M \times \mathbb{R}^{n+1,1}$.

1.2.1. Oriented conformal surfaces: generalities. Suppose M is provided with a conformal structure $\mathcal{C}$.

Remark 1.4. *The Levi-Civita connection is not a conformal invariant. In fact, (see, for example,* [**59**], §3.12*), given g, $g' := e^{2u} g \in \mathcal{C}$, for some $u \in C^\infty(M, \mathbb{R})$, the Levi-Civita connections ∇ and ∇', on (M,g) and (M,g'), respectively, are related by*

$$\nabla'_X Y = \nabla_X Y + (Xu)Y + (Yu)X - g(X,Y)(du)^*, \tag{1.5}$$

for all $X, Y \in \Gamma(TM)$, with $(du)^*$ denoting the contravariant form of du with respect to g, $(du)^* \in \Gamma(TM)$ defined by $g(X,(du)^*) = d_X u$, for all $X \in \Gamma(TM)$. The conformal variance of the Levi-Civita connection will, however, as we shall see, constitute, no limitation to the development of our theory of conformal submanifold geometry.

As an oriented conformal surface, M is canonically provided with a complex structure and the corresponding $(1,0)$- and $(0,1)$-decomposition. Unless indicated otherwise,

these shall be the underlying structures on $(M, \mathcal{C})$. In this section we recall these basic concepts and a few basic facts on Riemann surfaces.

The canonical complex structure. The fact that M is an oriented Riemannian surface enables us to refer to 90° rotations in the tangent spaces to M, a notion that is obviously invariant under conformal changes of the metric. We define then what we will refer to as the *canonical complex structure* on $(M, \mathcal{C})$, $J \in \Gamma(\mathrm{End}(TM))$, by 90° rotation in the positive direction. This terminology is not casual: J is an almost-complex structure, compatible with the conformal structure and, as it is well-known in the ambit of Riemannian geometry, parallel with respect to the connection induced in $\mathrm{End}(TM)$ by the Levi-Civita connection of (M, g), for each $g \in \mathcal{C}$; making (M, J) into a complex manifold, according to Newlander-Nirenberg Theorem.

The (1,0)- and (0,1)-decomposition. The almost-complex structure J gives rise to a decomposition

$$TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M, \tag{1.6}$$

for $T^{1,0}M$ and $T^{0,1}M$ the eigenspaces associated to i and $-i$, respectively, of the complex linear extension of J to $TM^{\mathbb{C}}$,

$$J = I \begin{cases} i & \text{on } T^{1,0}M \\ -i & \text{on } T^{0,1}M \end{cases} ;$$

that is,

$$T^{1,0}M = \langle X - iJX \rangle, \quad T^{0,1}M = \langle X + iJX \rangle,$$

fixing $X \in \Gamma(TM)$ locally never-zero. We refer to a section of $T^{1,0}M$ as a $(1,0)$-*vector field* and to a section of $T^{0,1}M$ as a $(0,1)$-*vector field*. The bundles $T^{1,0}M$ and $T^{0,1}M$ are maximal (with respect to the inclusion relation) isotropic subbundles of $TM^{\mathbb{C}}$. It is immediate and useful to observe that, in view of the parallelness of J,

$$\nabla\, \Gamma(T^{1,0}M) \subset \Omega^1(T^{1,0}M) \ , \ \nabla\, \Gamma(T^{0,1}M) \subset \Omega^1(T^{0,1}M), \tag{1.7}$$

for ∇ the Levi-Civita connection on (M, g), fixing $g \in \mathcal{C}$.

N.B.: The decomposition (1.6) is a particular case of a canonical decomposition, which is worth recalling: given V a non-degenerate complex 2-plane, V admits a unique decomposition $V = V_+ \oplus V_-$ into the direct sum of two null complex lines. Namely, fixing an orthonormal basis v_1, v_2 of V, the set of lines $\{V_+, V_-\}$ coincides with the set of lines $\{\langle v_1 + iv_2 \rangle, \langle v_1 - iv_2 \rangle\}$,

$$V = \langle v_1 + iv_2 \rangle \oplus \langle v_1 - iv_2 \rangle.$$

The non-degeneracy of V establishes $V_+ \cap (V_-)^\perp = \{0\}$. In the particular case V is real, we can choose the vectors v_1 and v_2 to be real, in which case the spaces V_+ and V_- are complex conjugate of each other.

Conformality. Given $X \in \Gamma(TM)$ never-zero, (X, JX) constitutes a direct orthogonal frame of TM with $(X,X) = (JX, JX)$, for all $(\,,\,) \in \mathcal{C}$. Hence a positive definite metric $(\,,\,)$ on M is in the conformal class $\mathcal{C}$ if and only if, fixing $X \in \Gamma(TM)$ locally never-zero,

$$(X,X) = (JX,JX), \quad (X,JX) = 0;$$

or, equivalently,

$$(Z,Z) = 0$$

(respectively, $(\bar{Z},\bar{Z}) = 0$), fixing $Z \in \Gamma(T^{1,0}M)$ (respectively,$\bar{Z} \in \Gamma(T^{0,1}M)$), locally never-zero. The conformality of a metric on M is equivalent to the isotropy of $T^{1,0}M$ or, equivalently, of $T^{0,1}M$, with respect to that metric. In particular, given an immersion ϕ of M into a Riemannian manifold and $i \neq j \in \{0,1\}$, the conformality of ϕ [with respect to any metric in $\mathcal{C}$], $g_\phi \in \mathcal{C}$, can be characterized by

$$(d^{i,j}\phi, d^{i,j}\phi) = 0,$$

fixing $i \neq j \in \{0,1\}$, where we use $(d^{i,j}\phi, d^{i,j}\phi)$ to denote $g_\phi|_{T^{i,j}M \times T^{i,j}M}$. In the paragraph below, a characterization of the conformality of ϕ in the light of a holomorphic chart of M is presented.

Holomorphicity. Let $z = x + iy : M \to \mathbb{C}$ be a chart of M. Throughout this text, let g_z denote the (positive definite) metric induced in M by $z : M \to (\mathbb{C}, \mathrm{Re}(\,,\,)_{\mathbb{C}})$,

$$g_z = dx^2 + dy^2.$$

We set, as usual,

$$\delta_x := dz^{-1}(1), \quad \delta_y := dz^{-1}(i),$$

defining an orthonormal frame δ_x, δ_y of TM with respect to g_z ((dx, dy) is the frame of $(TM)^*$ dual to (δ_x, δ_y)). The coordinates x, y are said to be *conformal coordinates* if the metric g_z is in the conformal class $\mathcal{C}$, or, equivalently,

$$J\delta_x = \pm \delta_y,$$

the chart $z : (M, J) \to \mathbb{C}$ is either holomorphic or anti-holomorphic, respectively, depending on the orientation on M.

Suppose z is a holomorphic chart of (M, J). In that case, $J\delta_x = \delta_y$, and, therefore,

$$\delta_z := \frac{1}{2}(\delta_x - i\delta_y), \quad \delta_{\bar{z}} := \frac{1}{2}(\delta_x + i\delta_y)$$

defines a never-zero $(1,0)$-vector field and, respectively, a never-zero $(0,1)$-vector field. Hence

$$T^{1,0}M = \langle \delta_z \rangle, \quad T^{0,1}M = \langle \delta_{\bar{z}} \rangle,$$

and, therefore,

$$TM^{\mathbb{C}} = \langle \delta_z, \delta_{\bar{z}} \rangle.$$

It is, perhaps, worth remarking that, as $(1,0)$- and $(0,1)$-vector fields, respectively, δ_z and $\delta_{\bar{z}}$ verify

$$J\delta_z = i\delta_z, \quad J\delta_{\bar{z}} = -i\delta_{\bar{z}}. \tag{1.8}$$

The fact that $\delta_z, \delta_{\bar{z}}$ constitutes a frame of $TM^{\mathbb{C}}$ with

$$[\delta_z, \delta_{\bar{z}}] = 0 \tag{1.9}$$

will be useful on many occasions.

Given a mapping η of M, we shall, alternatively, write η_x, η_y, η_z and $\eta_{\bar{z}}$ for, respectively, $d\eta(\delta_x)$, $d\eta(\delta_y)$, $d\eta(\delta_z)$ and $d\eta(\delta_{\bar{z}})$. In view of the holomorphicity of z, the conformality of a mapping η of M into a Riemannian manifold can be characterized by

$$(\eta_z, \eta_z) = 0,$$

or, equivalently,

$$(\eta_{\bar{z}}, \eta_{\bar{z}}) = 0;$$

whereas the holomorphicity of a map $\eta \in C^\infty(M, \mathbb{C})$ can be characterized by

$$\eta_{\bar{z}} = 0.$$

We complete this section with a basic remark on change of holomorphic coordinates. Let ω be another holomorphic chart of (M, J). The fact that $(dz,\, d\bar{z})$ is the frame of $((TM)^{\mathbb{C}})^*$ dual to $(\delta_z, \delta_{\bar{z}})$ makes clear that

$$d\omega = \omega_z \, dz$$

and, in particular, that the metric induced in M by ω relates to the one induced by z by

$$g_\omega = \mid \omega_z \mid^2 g_z. \tag{1.10}$$

The Hodge $*$-operator. Let P be a vector bundle over M, provided with a metric, and $p \in \{0,1,2\}$. Given $\mu \in \Omega^p(P)$ and $\eta \in \Omega^{2-p}(P)$, we define a 2-form $(\mu \wedge \eta) \in \Omega^2(\underline{\mathbb{C}})$ by, in the case $p = 1$,

$$(\mu \wedge \eta)_{(X,Y)} := (\mu_X, \eta_Y) - (\mu_Y, \eta_X)$$

and, in the case $p = 2$,

$$(\mu \wedge \eta)_{(X,Y)} := (\mu_{(X,Y)}, \eta),$$

for $X, Y \in \Gamma(TM)$; and, in the case $p = 0$, $(\mu \wedge \eta) := (\eta \wedge \mu)$. Note that, in the case $p = 1$, $(\mu \wedge \eta) = -(\eta \wedge \mu)$. Fix a metric $g \in \mathcal{C}$ and provide $L(TM, P)$ with the metric canonically induced by (TM, g) and P. Given $k \in \{0, 1, 2\}$, define a metric on $A^k(TM, P)$ by setting

$$(\xi_1 \wedge ... \wedge \xi_k, \gamma_1 \wedge ... \wedge \gamma_k) := \det((\xi_i, \gamma_j)),$$

for $\xi_i, \gamma_i \in \Gamma(L(TM, P))$. Recall the Hodge $*$-operator on p-forms over M, when provided with the metric g, transforming a form $\mu \in \Omega^p(P)$ into the form $*\mu \in \Omega^{2-p}(P)$ defined by the relation

$$(\mu \wedge \eta) = (*\mu, \eta)\, dA,$$

for all $\eta \in \Omega^{2-p}(P)$, with dA denoting the area element of (M, g). Recall that

$$* * \mu = (-1)^{p(2-p)+s} \mu,$$

for s the number of negative eigenvalues of the metric tensor of P. It is useful to observe that, in the particular case P is the trivial bundle $\mathrm{End}(\underline{\mathbb{R}}^{n+1,1}) := M \times \mathrm{End}(\mathbb{R}^{n+1,1})$, $s = 2(n + 1)$.

Lemma 1.5. *The Hodge $*$-operator on 1-forms over a surface is invariant under conformal changes of the metric on M.*

Before proceeding to the proof of the lemma, it is opportune and useful to observe how the volume element changes under conformal changes of the metric.

Lemma 1.6. *Let N be an n-dimensional oriented manifold and g and $g' := e^u g$, for some $u \in C^\infty(N, \mathbb{R})$, be conformally equivalent positive definite metrics in N. Then:*

$$\mathrm{dvol}_{(N,g')} = (e^u)^{\frac{n}{2}}\, \mathrm{dvol}_{(N,g)},$$

PROOF. Let $(X_i)_i$ be a direct orthonormal frame of (TN, g) and $(\xi_i)_i$ be the frame of $(TN)^*$ dual to $(X_i)_i$. Then

$$\mathrm{dvol}_{(N,g)} = \xi_1 \wedge ..... \wedge \xi_n.$$

On the other hand, clearly, $(\frac{1}{\sqrt{e^u}}\, X_i)_i$ is a direct orthonormal frame of (TN, g') with dual frame $(\sqrt{e^u}\, \xi_i)_i$, and, therefore, $\mathrm{dvol}_{(N,g')} = (\sqrt{e^u})^n\, \xi_1 \wedge ..... \wedge \xi_n$. □

Now we proceed to the proof of Lemma 1.5.

PROOF. Let g_1 and $g_2 := e^u g_1$, for some $u \in C^\infty(M, \mathbb{R})$, be conformally equivalent positive definite metrics in M. For $i = 1, 2$, let $*_i$ and dA_i denote, respectively, the Hodge $*$-operator and the area element of (M, g_i), and $(\,,)_i$ denote the Hilbert-Schmidt

metric in $L((TM, g_i), P)$. Fix $\mu, \eta \in \Omega^1(P)$. The proof will consist of showing that

$$(\mu \wedge \eta) = (*_2\, \mu, \eta)_1\, dA_1. \tag{1.11}$$

By definition of $*_2$, and according to Lemma 1.6, $(\mu \wedge \eta) = (*_2\, \mu, \eta)_2\, e^u dA_1$. On the other hand, given an orthonormal frame X_1, X_2 of (TM, g_1), $\frac{1}{\sqrt{e^u}}\, X_1, \frac{1}{\sqrt{e^u}}\, X_2$ is an orthonormal frame of (TM, g_2) and, therefore, $(*_2\, \mu, \eta)_2 = \frac{1}{e^u}\, (*_2\, \mu, \eta)_1$, completing the proof. □

The Hodge $*$-operator on 1-forms over $(M, \mathcal{C})$ is closely related to the canonical complex structure in $(M, \mathcal{C})$ in a well-known result in the ambit of Riemannian geometry: $*$ acts on forms $\omega \in \Omega^1(P)$ by

$$*\, \omega = -\omega J. \tag{1.12}$$

(1,0)-,(0,1)- and (1,1)-forms. The decomposition (1.6) provides the (standard) decomposition of a 1-form into its $(1, 0)$ and $(0, 1)$ parts, as well as that of a 2-form into its $(2, 0)$, $(1, 1)$ and $(0, 2)$ parts. Recall that there are no non-zero $(2, 0)$- or $(0, 2)$-forms on a surface.

Let P be a vector bundle over M, provided with a metric, and ξ be a 1-form with values in P. Since $\xi^{1,0}$ and $\xi^{0,1}$ are the complexifications of, respectively, a complex linear and a complex anti-linear sections of $\mathrm{Hom}(TM, P)$, we have, by equation (1.12),

$$*\, \xi^{1,0} = -i\, \xi^{1,0}, \quad *\, \xi^{0,1} = i\, \xi^{0,1},$$

so that $*\, \xi = -i(\xi^{1,0} - \xi^{0,1})$ and, therefore,

$$\xi^{1,0} = \frac{1}{2}\, (\xi + i *\, \xi), \quad \xi^{0,1} = \frac{1}{2}\, (\xi - i *\, \xi),$$

having in consideration that $\overline{*\,\xi} = *\, \overline{\xi}$. Note that $\overline{\xi^{1,0}} = \overline{\xi}^{\,0,1}$. In particular, ξ is real ($\overline{\xi} = \xi$) if and only if $\xi^{0,1} = \overline{\xi^{1,0}}$.

1.2.2. Conformal immersions of surfaces in the projectivized light-cone. Having modeled the conformal n-sphere on the projectivized light-cone of $\mathbb{R}^{n+1,1}$, and, in this way, all n-dimensional space-forms on submanifolds of $\mathbb{P}(\mathcal{L})$, we approach a surface conformally immersed in a space-form as a null line bundle Λ defining an immersion of an oriented surface, which we provide with the conformal structure induced by Λ, into the projectivized light-cone. In this work, we restrict to surfaces in S^n which are not contained in any subsphere of S^n. Such a surface defines a surface in any given space-form, by means of a lift, whose study is Möbius equivalent to the study of the surface and which will be often considered. Namely, given $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero, we have, locally, $(\sigma, v_\infty) \neq 0$, and Λ defines then a local immersion $\sigma_\infty := (\pi|_{S_{v_\infty}})^{-1} \circ \Lambda = \frac{-1}{(\sigma, v_\infty)}\, \sigma : M \to S_{v_\infty}$, of M into the space-form S_{v_∞}.

Let Λ be a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$. In particular, Λ defines a smooth map $\Lambda : M \to \mathbb{P}(\mathcal{L})$, by assigning to each p in M the corresponding fibre, Λ_p. If Λ is an immersion, then Λ induces naturally, from the conformal structure on $\mathbb{P}(\mathcal{L})$, a conformal structure in M, which we denote by $\mathcal{C}_\Lambda$, making

$$\Lambda : (M, \mathcal{C}_\Lambda) \to \mathbb{P}(\mathcal{L})$$

into a conformal immersion of M into the projectivized light-cone.

Proposition 1.7. *If $\Lambda : M \to \mathbb{P}(\mathcal{L})$ is an immersion, then, given a never-zero section σ of Λ, the metric g_σ induced in M by $\sigma : M \to \mathbb{R}^{n+1,1}$ is in the conformal class $\mathcal{C}_\Lambda$ of metrics,*

$$g_\sigma \in \mathcal{C}_\Lambda.$$

Proof. For t_0 unit time-like vector, $\langle t_0 \rangle^\perp$ is an Euclidean space and, therefore, (σ, t_0) is never-zero. Furthermore, $(\pi_{\mathcal{L}}|_{S_{t_0}})^{-1}\Lambda = -(\sigma, t_0)^{-1}\,\sigma$, and, therefore,

$$d((\pi_{\mathcal{L}}|_{S_{t_0}})^{-1}\,\Lambda) = \frac{(d\sigma, t_0)}{(\sigma, t_0)^2}\,\sigma - \frac{1}{(\sigma, t_0)}\,d\sigma.$$

As $(\sigma, \sigma) = 0 = (\sigma, d\sigma)$, we conclude that, for $X, Y \in \Gamma(TM)$,

$$(d_X((\pi_{\mathcal{L}}|_{S_{t_0}})^{-1}\,\Lambda), d_Y((\pi_{\mathcal{L}}|_{S_{t_0}})^{-1}\,\Lambda)) = \frac{1}{(\sigma, t_0)^2}\,(d_X\sigma, d_Y\sigma)$$

or, equivalently,

$$g_\sigma(X, Y) = (\sigma, t_0)^2 (d_X\Lambda, d_Y\Lambda)_{t_0} = (\sigma, t_0)^2 g_\Lambda^{t_0}(X, Y),$$

for $(\, , \,)_{t_0}$ the (positive definite) metric induced in $\mathbb{P}(\mathcal{L})$ by $(\pi_{\mathcal{L}}|_{S_{t_0}})^{-1}$ and $g_\Lambda^{t_0}$ the (positive definite) metric induced in M by Λ from $(\, , \,)_{t_0}$. We conclude that g_σ is a positive definite metric conformally equivalent to $g_\Lambda^{t_0} \in \mathcal{C}_\Lambda$. $\square$

Given σ a never-zero section of Λ, $\Lambda = \pi_{\mathcal{L}}\,\sigma$ and, therefore, $d\Lambda = d(\pi_{\mathcal{L}})_\sigma\,d\sigma$. Hence, if Λ is an immersion, then so is σ, and, conversely, if σ is an immersion, then, in view of (1.2), Λ is an immersion if and only if Λ is in direct sum with $d\sigma(TM)$. Set

$$\Lambda^{(1)} := \langle \sigma, d\sigma(e_1), d\sigma(e_2) \rangle = \Lambda + d\sigma(TM),$$

independently of the choices of a never-zero section σ of Λ and of a frame e_1, e_2 of TM. This is, indeed, well-defined: given a never-zero $\lambda \in \Gamma(\underline{\mathbb{R}})$,

$$d_X(\lambda\sigma) = \lambda d_X\sigma + (d_X\lambda)\sigma = \lambda d_X\sigma \bmod \langle \sigma \rangle.$$

Proposition 1.8. $\Lambda : M \to \mathbb{P}(\mathcal{L})$ *is an immersion if and only if* $\operatorname{rank} \Lambda^{(1)} = 3$.

Proof. Fix σ a never-zero section of Λ. If $\operatorname{rank} \Lambda^{(1)} = 3$, then $\operatorname{rank} d\sigma(TM) = 2$, or, equivalently, σ is an immersion, and $d\sigma(TM)$ is in direct sum with Λ. Hence Λ is an immersion.

Conversely, if Λ is an immersion, then so is σ, in which case $d\sigma(TM)$ is a bundle of Euclidean 2-planes and, therefore, $\operatorname{rank}\Lambda^{(1)} = 3$. $\square$

In the case $\Lambda : M \to \mathbb{P}(\mathcal{L})$ is an immersion, $\Lambda(M)$ consists of a surface [conformally immersed] in $\mathbb{P}(\mathcal{L})$,[3] which we refer to, alternatively, as the surface Λ. We restrict ourselves to surfaces Λ in S^n for which

$$\Lambda(M) \not\subseteq S^k, \tag{1.13}$$

for all $k < n$, and, in particular, to surfaces which do not lie in any 2-sphere. This ensures, in particular, that, for a general non-zero $v_\infty \in \mathbb{R}^{n+1,1}$, given $\sigma \in \Gamma(\Lambda)$ never-zero, we have, locally, $(\sigma, v_\infty) \neq 0$. In fact, if $(\sigma_p, v_\infty) = 0$ for all $p \in M$, then $\Lambda(M) \subset \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$, which, in the case v_∞ is time-light or light-like, is impossible, according to Lemma 1.2, and, in the case v_∞ is space-like, contradicts (1.13) (in the case v_∞ is space-like, $\mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$ is an hypersphere in S^n).

A surface Λ defines a local immersion of M into $\mathbb{P}(\mathcal{L}) \backslash \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$ and then, via the diffeomorphism $\pi_{\mathcal{L}}|_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L}) \backslash \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$, a local immersion

$$\sigma_\infty := (\pi_{\mathcal{L}}|_{S_{v_\infty}})^{-1} \circ \Lambda = \frac{-1}{(\sigma, v_\infty)}\, \sigma : M \to S_{v_\infty},$$

of M into the space-form S_{v_∞}, which we refer to as *the surface defined by* Λ *in* S_{v_∞}. The surfaces σ_∞, with $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero, defined by Λ, form a family of Möbius equivalent surfaces, whose study is Möbius equivalent to the study of the surface Λ.

[3]If $\Lambda : M \to \mathbb{P}(\mathcal{L})$ is an immersion, then Λ defines a surface in spherical n-space. Since $n \geq 3$, we can choose $v_\infty \in \mathcal{L}$ such that $\Lambda_p \neq \langle v_\infty \rangle$,

$$\Lambda_p \in \mathbb{P}(\mathcal{L}) \backslash \{\langle v_\infty \rangle\} \cong S^n \backslash \{x_0\},$$

for all $p \in M$, showing that Λ defines, furthermore, a surface in Euclidean n-space. In the case M is compact, $\Lambda(M) \subset S^n$ is a compact 2-dimensional manifold and, therefore, $S^n \backslash \Lambda(M)$ is a non-empty open subset. This shows that $\Lambda(M)$ avoids some hypersphere in S^n, or, equivalently, that we can choose a space-like vector v_∞ for which $\Lambda_p \not\subseteq \langle v_\infty \rangle^\perp, \forall p \in M$. For such a v_∞, Λ defines a surface in $\mathbb{P}(\mathcal{L}) \backslash \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$ and, locally (on a connected component of M), in hyperbolic n-space.

CHAPTER 2

The central sphere congruence

Following [**14**], we introduce the central sphere congruence, a fundamental construction of Möbius invariant surface geometry, which will be basic to our study of surfaces. The concept has its origins in the nineteenth century with the introduction of the mean curvature sphere of a surface at a point, by S. Germain [**30**]. The terminology reflects the central role played by the mean curvature of a surface. By the turn of the century, the family of the mean curvature spheres of a surface was known as the central sphere congruence, cf. W. Blaschke [**4**]. Nowadays, after R. Bryant's paper [**9**], it goes as well by the name conformal Gauss map.

We start by recalling some basic concepts in Riemannian geometry. Given a Riemannian manifold $\bar{M}$ and an immersion $f : M \to \bar{M}$, the pull-back bundle $f^*T\bar{M}$ splits into the direct sum $T_f \oplus N_f$, where T_f and N_f denote, respectively, the tangent bundle, $df(TM)$, and the normal bundle, $(df(TM))^\perp$, to f. Let π_{T_f} and π_{N_f} denote the orthogonal projections of $f^*T\bar{M}$ onto T_f and N_f, respectively. The normal bundle is provided with the connection

$$\nabla^{N_f} = \pi_{N_f} \circ \nabla^{f^*T\bar{M}}|_{\Gamma(N_f)},$$

where $\nabla^{f^*T\bar{M}}$ denotes the pull-back connection by f of the Levi-Civita connection on $T\bar{M}$. Recall the second fundamental form of f, $\Pi \in \Gamma(L^2(TM, N_f))$, defined by

$$\Pi(X,Y) = \pi_{N_f}(\nabla_X^{f^*T\bar{M}} d_Y f),$$

and the mean curvature vector of f (or of $f(M) \subset \bar{M}$, the surface M immersed in $\bar{M}$ by f),

$$\mathcal{H} = \frac{1}{2}\,\mathrm{tr}_{g_f}\Pi \in \Gamma(N_f),$$

with tr_{g_f} indicating trace computed with respect to g_f. Recall that

$$\nabla_X^{f^*T\bar{M}} d_Y f - \nabla_Y^{f^*T\bar{M}} d_X f = d_{[X,Y]} f,$$

for all $X, Y \in \Gamma(TM)$, establishing, in particular, the symmetry of Π. Recall the fundamental equation in Riemannian Geometry:

$$\pi_{T_f}(\nabla_X^{f^*T\bar{M}} df(Y)) = df(\nabla_X^{TM} Y), \tag{2.1}$$

for ∇^{TM} the Levi-Civita connection of M when provided with the metric induced by f from the one on $\bar{M}$; for all $X, Y \in \Gamma(TM)$. Fix a unit $\xi \in \Gamma(N_f)$ and recall the shape operator $A^\xi \in \Gamma(\mathrm{End}\,(TM, T_f))$, of f with respect to ξ, given by

$$A^\xi(X) = -\pi_{T_f}(\nabla_X^{f^*T\bar{M}}\xi),$$

and the mean curvature of f (or of $f(M)$), with respect to ξ,

$$H^\xi = \frac{1}{2}\,\mathrm{tr}A^\xi \in \Gamma(\underline{\mathbb{R}}).$$

For an isometric immersion, the second fundamental form and the shape operator with respect to ξ are related by

$$(\Pi(X,Y),\xi) = (A^\xi(X), Y). \tag{2.2}$$

In particular, for an isometric immersion,

$$H^\xi = (\mathcal{H}, \xi).$$

Equation (2.2) establishes, on the other hand, the symmetry - and consequent diagonalizability - of the shape operator of an isometric immersion. The shape operators A^ξ and $A^{-\xi}$ are symmetrical and, therefore, share eigenspaces and have symmetrical eigenvalues. Recall that, in the case f is an isometric immersion of M into $\bar{M} = \mathbb{R}^3$, the eigenvalues k_1 and k_2 of A^ξ are called the principal curvatures of f, defined up to sign; that a point p in M at which the principal curvatures are the same is said to be umbilic; and that, away from umbilic points, the directions defined by the two lines consisting of the common eigenspaces of A^ξ and $A^{-\xi}$ are called the principal directions of f, whilst for umbilic points all directions are said to be principal. In particular, the mean curvature of $f : M \to \mathbb{R}^3$ with respect to either of the two unit normal vector fields to f is given, up to sign, by

$$H = \frac{k_1 + k_2,}{2}, \tag{2.3}$$

the arithmetic mean of the principal curvatures. It is opportune to recall that, in the case f defines an isometric immersion of M in Euclidean 3-space, the Gaussian curvature of the surface $f(M)$, or, equally (cf. *theorema egregium* of Gauss), that of M, can be obtained as

$$K = \det A^\xi = k_1 k_2, \tag{2.4}$$

fixing ξ a unit normal vector field to f. For that, and for further reference, recall the Gauss equation,

$$\begin{aligned} &R(X,Y,Z,W) - \bar{R}(df(X), df(Y), df(Z), df(W)) \\ &= (\Pi(Y,W), \Pi(X,Z)) - (\Pi(X,W), \Pi(Y,Z)), \end{aligned}$$

for $X, Y, Z, W \in \Gamma(TM)$; relating the curvature tensors R and $\bar{R}$ of (M, g_f) and $\bar{M}$, respectively. It establishes, in particular, that, if $\bar{M}$ has constant sectional curvature $\bar{K} = \bar{K}(x)$, for $x \in \bar{M}$, then the Gaussian curvature K of M relates to $\bar{K}$ by

$$K - \bar{K} = (\Pi(X_1, X_1), \Pi(X_2, X_2)) - (\Pi(X_1, X_2), \Pi(X_1, X_2)), \tag{2.5}$$

fixing an orthonormal frame X_1, X_2 of (TM, g_f). Now consider the particular case $\bar{M} = \mathbb{R}^3$, fix $\xi \in \Gamma(N_f)$ unit and consider X_1, X_2 to be a frame along principal directions of f, say $A^\xi X_i = k_i X_i$, for $i = 1, 2$, (whose existence is established by the symmetry of A^ξ). Then, according to (2.2), $\Pi(X_i, X_i) = k_i \xi$, for $i = 1, 2$, and $\Pi(X_1, X_2) = 0$ and the conclusion follows then from (2.5).

Remark 2.1. *Suppose g and g' are conformally equivalent positive definite metrics on $\bar{M}$, say $g' = e^{2u} g$, for some $u \in C^\infty(\bar{M}, \mathbb{R})$. Following* (1.5)*, we get that the connections ∇ and ∇', induced in the pull-back bundle $f^* T\bar{M}$ by the Levi-Civita connections on $(\bar{M}, g)$ and $(\bar{M}, g')$, respectively, are related by*

$$\nabla'_X Y = \nabla_X Y + ((f^* du) Y) df(X) + ((f^* du) df(X)) Y - g(df(X), Y) f^*(du)^*,$$

for $X \in \Gamma(TM)$, $Y \in \Gamma(f^* T\bar{M})$, $f^* du$ the pull-back by f of du,

$$(f^* du) Z := (x \mapsto du_{f(x)}(Z_x)) \in C^\infty(M, \mathbb{R}),$$

given $Z \in \Gamma(f^* T\bar{M})$; and, analogously, $f^*(du)^*$ the pull-back by f of $(du)^*$. It follows that the second fundamental forms Π and Π' of the immersions by f of M into $(\bar{M}, g)$ and $(\bar{M}, g')$, respectively, are related by

$$\Pi'(X, Y) = \Pi(X, Y) - g_f(X, Y)\, \pi_{N_f}(f^*(du)^*), \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$, with g_f denoting the metric induced in M by f from the metric g. Consequently,

$$\mathcal{H}' = e^{-2u \circ f} \mathcal{H} - e^{-2u \circ f} \pi_{N_f}(f^*(du)^*), \tag{2.7}$$

relating the mean curvature vectors $\mathcal{H}'$ and $\mathcal{H}$ of $f : M \to (\bar{M}, g)$ and $f : M \to (\bar{M}, g')$, respectively. The conformal variance of the mean curvature vector will, however, constitute no limitation to the development of our conformal submanifold theory, as we shall see.

Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone.

2.1. Central sphere congruence and mean curvature

Bryant [**9**] established the existence of a congruence of 2-spheres, named the conformal Gauss map, tangent to a Riemann surface isometrically immersed in S^3, sharing the mean curvature at each point with the surface. This congruence of spheres can be generalized to surfaces conformally immersed in S^n by means of the central sphere

congruence, which we present in this section, following [**14**].

A bundle $V \subset \underline{\mathbb{R}}^{n+1,1}$ of $(3,1)$-planes is said to be an *enveloping* 2-*sphere congruence* of the surface Λ if the 2-spheres $\mathbb{P}(\mathcal{L} \cap V_p) \subset \mathbb{P}(\mathcal{L}) \cong S^n$, for $p \in M$, have first order contact with Λ, i.e., $\Lambda^{(1)} \subset V$.

Definition 2.2. *We define an enveloping* 2-*sphere congruence to* Λ*, said to be the* central sphere congruence*, by*

$$S := \langle \sigma, d_{e_1}\sigma, d_{e_2}\sigma, \sum_i d_{e_i} d_{e_i}\sigma \rangle \subset \underline{\mathbb{R}}^{n+1,1},$$

independently of the choices of a never-zero $\sigma \in \Gamma(\Lambda)$ *and of a local orthonormal frame* $(e_i)_i$ *of* TM *with respect to the metric* g_σ*. We may, alternatively, use the notation* S_Λ *for* S*.*

We shall now recognize that this is, in fact, well-defined. Fix σ a never-zero section of Λ and $(e_i)_i$ an orthonormal frame of (TM, g_σ). First of all, observe that, by differentiating $(\sigma, d_{e_i}\sigma) = 0$ we get $(\sigma, d_{e_i} d_{e_i}\sigma) = -(d_{e_i}\sigma, d_{e_i}\sigma) = -1$, for $i = 1, 2$, and, consequently,

$$(\sigma, \sum_i d_{e_i} d_{e_i}\sigma) = -2, \tag{2.8}$$

which, together with $(\sigma, \sigma) = 0 = (\sigma, d\sigma)$, shows that $\sum_i d_{e_i} d_{e_i}\sigma$ is not a section of $\Lambda^{(1)}$. Thus $S = \Lambda^{(1)} \oplus \langle \sum_i d_{e_i} d_{e_i}\sigma \rangle$. Let us now show that $\Lambda^{(1)} \oplus \langle \sum_i d_{e_i} d_{e_i}\sigma \rangle$ does not depend on the choices of σ and $(e_i)_i$. Consider the Hessian of $\sigma : (M, g_\sigma) \to \mathbb{R}^{n+1,1}$, the section $\nabla d\sigma$ of $S^2(TM, \underline{\mathbb{R}}^{n+1,1})$, given by $\nabla d\sigma(X, Y) = d_X d_Y \sigma - d\sigma(\nabla^\sigma_X Y)$, for ∇^σ the Levi-Civita connection on (M, g_σ). Writing tr_{g_σ} for the trace with respect to the metric g_σ, we have

$$\sum_i d_{e_i} d_{e_i}\sigma = \mathrm{tr}_{g_\sigma} \nabla d\sigma + d\sigma(\sum_i \nabla^\sigma_{e_i} e_i) = (\mathrm{tr}_{g_\sigma} \nabla d\sigma) \bmod \Lambda^{(1)},$$

showing that $\Lambda^{(1)} \oplus \langle \sum_i d_{e_i} d_{e_i}\sigma \rangle$ does not depend on the choice of $(e_i)_i$. On the other hand, given $\lambda \in \Gamma(\underline{\mathbb{R}})$ never-zero, the metrics induced by σ and $\sigma' := \lambda\sigma$ are related by $g_{\sigma'} = \lambda^2 g_\sigma$, so that $(\lambda^{-1} e_i)_i$ constitutes an orthonormal frame of $(TM, g_{\sigma'})$. Now

$$\begin{aligned} \sum_i d_{\lambda^{-1}e_i} d_{\lambda^{-1}e_i}\sigma' &= \lambda^{-1} \sum_i d_{e_i} d_{e_i}\sigma + \lambda^{-2} \sum_i ((d_{e_i}\lambda)(d_{e_i}\sigma) + (d_{e_i} d_{e_i}\lambda)\sigma) \\ &= (\lambda^{-1} \sum_i d_{e_i} d_{e_i}\sigma) \bmod \Lambda^{(1)} \end{aligned}$$

shows that

$$\Lambda^{(1)} \oplus \langle \sum_i d_{e_i} d_{e_i}\sigma \rangle = \Lambda^{(1)} \oplus \langle \sum_i d_{\lambda^{-1}e_i} d_{\lambda^{-1}e_i}\sigma' \rangle,$$

showing the independence of $\Lambda^{(1)} \oplus \langle \sum_i d_{e_i} d_{e_i}\sigma \rangle$ with respect to the choice of σ.

To recognize that S consists of a bundle of $(3,1)$-spaces in $\mathbb{R}^{n+1,1}$, we just need to verify that it is a non-degenerate rank 4 bundle, for $\sigma \in \Gamma(S)$ is light-like. The fact that Λ is an immersion gives $\operatorname{rank} \Lambda^{(1)} = 3$, whereas, by equation (2.8), $\operatorname{rank} \langle \sum_i d_{e_i} d_{e_i} \sigma \rangle = 1$. Thus $\operatorname{rank} S = 4$. On the other hand, given $i, j \in \{1, 2\}$,

$$\begin{aligned}(d_{e_j}\sigma, d_{e_i} d_{e_i}\sigma) &= d_{e_i}(d_{e_j}\sigma, d_{e_i}\sigma) - (d_{e_i} d_{e_j}\sigma, d_{e_i}\sigma) \\ &= d_{e_i}\delta_{i,j} - (d_{e_j} d_{e_i}\sigma, d_{e_i}\sigma) \\ &= -\frac{1}{2}\, d_{e_j}(d_{e_i}\sigma, d_{e_i}\sigma) \\ &= 0,\end{aligned}$$

and, therefore, $(d_{e_j}\sigma, \sum_k d_{e_k} d_{e_k}\sigma) = 0$. It follows that the matrix of the metric on S in the frame $\sigma, d_{e_1}\sigma, d_{e_2}\sigma, \sum_i d_{e_i} d_{e_i}\sigma$ is

$$\begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & a \end{pmatrix}$$

for some $a \in \mathbb{R}$, whose determinant is -4, which shows that S is non-degenerate.

Lastly, and explicitly, we have $\Lambda^{(1)} \subset S$, which completes this verification.

Remark 2.3. *Observe from the previous verification that the definition of S is, furthermore, independent of conformal changes of the metric g_σ. In particular, given a holomorphic chart $z = x + iy$ of $(M, \mathcal{C}_\Lambda)$, we have*

$$S = \langle \sigma, \sigma_x, \sigma_y, \sigma_{xx} + \sigma_{yy} \rangle.$$

Note that $\sigma_{xx} + \sigma_{yy} = 4\sigma_{z\bar{z}}$. It follows that the complexification of the central sphere congruence of Λ is given by

$$S = \langle \sigma, \sigma_z, \sigma_{\bar{z}}, \sigma_{z\bar{z}} \rangle \subset (\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}.$$

Although the mean curvature vector is not a conformal invariant (cf. Remark 2.1), under a conformal change of the metric on S^n, it changes in the same way for the surface M immersed in S^n by Λ and each 2-sphere $\mathbb{P}(\mathcal{L} \cap S_p)$, with $p \in M$, canonically immersed in S^n. The condition $\sigma_{z\bar{z}} \in \Gamma(S)$ establishes that, at each point, the surface Λ and the sphere $\mathbb{P}(\mathcal{L} \cap S_p)$ share the same mean curvature vector. S is, in this way, a congruence of *osculating* spheres to the surface Λ: for each $p \in M$, $\mathbb{P}(\mathcal{L} \cap S_p)$ is the unique 2-sphere in S^n tangent to $\Lambda(M)$ at $\Lambda(p)$ whose mean curvature vector at $\Lambda(p)$ is the mean curvature vector of M at p.

The central sphere congruence of Λ defines naturally a map

$$S : M \to \mathcal{G} := Gr_{(3,1)}(\mathbb{R}^{n+1,1})$$

into the connected component of the Grassmannian of order 4 of $\mathbb{R}^{n+1,1}$ constituted by the subspaces with signature $(3,1)$. Throughout this text, let π_S and $\pi_{S^\perp}$ denote the orthogonal projections of $\underline{\mathbb{R}}^{n+1,1}$ onto S and $S^\perp$, respectively. Let T and $T^\perp$ be the bundles over $\mathcal{G}$ whose fibres at each $V \in \mathcal{G}$ are, respectively, V and $V^\perp$. The tangent bundle to $\mathcal{G}$ at T is identified with the bundle $\mathrm{Hom}(T, T^\perp)$ via the bundle isomorphism given by

$$X \mapsto (\rho \mapsto \pi_{T^\perp}(d_X \rho)), \tag{2.9}$$

for $\pi_{T^\perp}$ the orthogonal projection of $\mathbb{R}^{n+1,1}$ onto $V^\perp$. This is indeed well-defined, as, given $\rho \in \Gamma(T)$, $(\rho \mapsto \pi_{T^\perp}(d_X \rho))$ is tensorial. This identification makes $\mathcal{G}$ into a pseudo-Riemannian manifold.[1] Under this identification, the pull-back bundle $S^*T\mathcal{G}$ is identified with the pull-back by S of $\mathrm{Hom}(T, T^\perp)$,

$$S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp).$$

Proposition 2.4. *The central sphere congruence of Λ is conformal with respect to the conformal structure $\mathcal{C}_\Lambda$ induced in M by Λ.*

Before we proceed to the proof of the proposition, fix σ a never-zero section of Λ and z a holomorphic chart of $(M, \mathcal{C}_\Lambda)$ and let us spend some moments contemplating the orthogonality relations of the frame $\{\sigma, \sigma_z, \sigma_{\bar{z}}, \sigma_{z\bar{z}}\}$ of S, beyond the very well-known $(\sigma, \sigma) = 0$ and subsequent

$$(\sigma, \sigma_z) = 0 = (\sigma, \sigma_{\bar{z}}).$$

The conformality of $\sigma : (M, \mathcal{C}_\Lambda) \to \mathbb{R}^{n+1,1}$ gives

$$(\sigma_z, \sigma_z) = 0 = (\sigma_{\bar{z}}, \sigma_{\bar{z}}),$$

and differentiation shows then that

$$(\sigma_z, \sigma_{z\bar{z}}) = 0 = (\sigma_{\bar{z}}, \sigma_{z\bar{z}}).$$

The fact that g_σ is conformally equivalent to g_z ensures that

$$(\sigma_z, \sigma_{\bar{z}}) = \frac{1}{4}\left(g_\sigma(\delta_x, \delta_x) + g_\sigma(\delta_y, \delta_y)\right) \tag{2.10}$$

is never-zero, whilst differentiation of $(\sigma, \sigma_z) = 0$ shows that

$$(\sigma, \sigma_{z\bar{z}}) = -(\sigma_z, \sigma_{\bar{z}}).$$

As for $(\sigma_{z\bar{z}}, \sigma_{z\bar{z}})$, nothing can be ensured in a general situation. As a final remark, we introduce a specific choice of a never-zero section of Λ, in relation to the chart z, that provides an extra specificity on the orthogonality relations presented above and that will, for that reason, be useful in some moments in the future. As we know, each

[1]This identification is the particular case $\mathcal{G} = Gr_{(3,1)}(\mathbb{R}^{n+1,1})$ of a standard procedure to induce a pseudo-Riemannian structure in a general Grassmannian $\mathcal{G} = Gr_{(r,s)}(\mathbb{R}^{p,q})$.

never-zero section of Λ induces in M a metric conformally equivalent to the metric g_z induced by z. If we make a choice of one of the two components of the light-cone, say $\mathcal{L}^+$, then there is a unique section $\sigma^z : M \to \mathcal{L}^+$ of Λ whose induced metric coincides with the metric induced by z,

$$g_{\sigma^z} = g_z.$$

We refer to σ^z as the *normalized section* of Λ with respect to z. According to (2.10), $(\sigma^z_z, \sigma^z_{\bar{z}})$ is constantly equal to $\frac{1}{2}$,

$$(\sigma^z_z, \sigma^z_{\bar{z}}) = \frac{1}{2}.$$

In particular, $(\sigma^z_z, \sigma^z_{\bar{z}})_z = 0 = (\sigma^z_z, \sigma^z_{\bar{z}})_{\bar{z}}$ or, equivalently,

$$(\sigma^z_z, \sigma^z_{\bar{z}\bar{z}}) = 0.$$

Now we proceed to the proof of Proposition 2.4.

PROOF. Fix a holomorphic chart $z = x + iy$ of $(M, \mathcal{C}_\Lambda)$. The proof will consist of showing that $(S_z, S_z) = 0$, or, equivalently, that $\mathrm{tr}(S^t_z\, S_z) = 0$.

Fix a never-zero section σ of Λ. According to the orthogonality relations of the frame $\sigma, \sigma_z, \sigma_{\bar{z}}, \sigma_{z\bar{z}}$ of S, we have $\langle \sigma, \sigma_{\bar{z}} \rangle^\perp \cap S = \langle \sigma, \sigma_{\bar{z}} \rangle$. On the other hand,

$$S_z(\sigma) = \pi_{S^\perp}(\sigma_z) = 0 = \pi_{S^\perp}(\sigma_{\bar{z}z}) = S_z(\sigma_{\bar{z}}), \tag{2.11}$$

and, therefore, $\langle \sigma, \sigma_{\bar{z}} \rangle \subset \ker S_z$. It follows that $\mathrm{Im}\, S^t_z \subset (\ker S_z)^\perp \cap S \subset \langle \sigma, \sigma_{\bar{z}} \rangle$ and, consequently, by (2.11), that

$$S_z\, S^t_z = 0,$$

which completes the proof. $\square$

It was Bryant [**9**] who established the existence of a congruence of 2-spheres tangent to a Riemann surface isometrically immersed in S^3 and with the same mean curvature as the surface at each point. Bryant named it the conformal Gauss map. This justifies the alternative terminology, after Bryant's paper [**9**], of conformal Gauss map for the central sphere congruence, although the central sphere congruence carries, not only first order contact information, but second order as well.

2.2. The normal bundle to the central sphere congruence

The bundle $S^\perp$, normal to the central sphere congruence of Λ, can be identified with the normal bundle to Λ, when regarded as a surface in a space-form, via an isometric isomorphism of bundles with connections, as we present next, following [**14**].

We provide S and $S^\perp$ with the connections ∇^S and $\nabla^{S^\perp}$, respectively, defined by orthogonal projection of the trivial flat connection d on $\underline{\mathbb{R}}^{n+1,1}$,

$$\nabla^S := \pi_S \circ d|_{\Gamma(S)}, \quad \nabla^{S^\perp} := \pi_{S^\perp} \circ d|_{\Gamma(S^\perp)},$$

which we immediately verify to be metric connections.

Fix a non-zero v_∞ in $\mathbb{R}^{n+1,1}$ and consider the surface $\sigma_\infty : M \to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. For simplicity, we write g_∞, rather than g_{σ_∞}, for the metric induced in M by σ_∞, as well as N_∞ for the normal bundle to σ_∞. Let Π_∞ and $\mathcal{H}_\infty$ denote, respectively, the second fundamental form and the mean curvature vector of σ_∞. We shall keep this notation throughout this text.

Proposition 2.5. *The normal bundle N_∞ to σ_∞ is identified with the bundle $S^\perp$ normal to the central sphere congruence of Λ,*

$$N_\infty \cong S^\perp,$$

by the isomorphism

$$\mathcal{Q} : N_\infty \to S^\perp$$

of bundles provided with a metric and a connection defined by

$$\xi \mapsto \xi + (\xi, \mathcal{H}_\infty)\sigma_\infty.$$

Proof. The pull-back bundle by σ_∞ of the tangent bundle TS_{v_∞} consists of the orthogonal complement in $\underline{\mathbb{R}}^{n+1,1}$ of the non-degenerate bundle $\langle \sigma_\infty, v_\infty \rangle$ (cf. section 1.1). Let π_{N_∞} denote the orthogonal projection of

$$\underline{\mathbb{R}}^{n+1,1} = d\sigma_\infty(TM) \oplus N_\infty \oplus \langle v_\infty, \sigma_\infty \rangle$$

onto N_∞. Since the metric in S_{v_∞} is the one inherited from $\mathbb{R}^{n+1,1}$, Π_∞ is simply given by

$$\Pi_\infty(X, Y) = \pi_{N_\infty}(d_X d_Y \sigma_\infty),$$

for $X, Y \in \Gamma(TM)$, so that, for $\xi \in \Gamma(N_\infty)$ and $(e_i)_i$ an orthonormal frame of (TM, g_∞), $(\xi, \sum_i d_{e_i} d_{e_i} \sigma_\infty) = 2(\xi, \mathcal{H}_\infty)$ and, therefore, by (2.8), $(\xi + (\xi, \mathcal{H}_\infty)\sigma_\infty, \sum_i d_{e_i} d_{e_i} \sigma_\infty) = 0$. Together with the fact that

$$N_\infty \subset \sigma_\infty^* TS_{v_\infty} = \langle \sigma_\infty, v_\infty \rangle^\perp,$$

this shows that $\xi + (\xi, \mathcal{H}_\infty)\sigma_\infty$ is, in fact, a section of $S^\perp$.

Clearly, $\mathcal{Q}$ is isometric, and, therefore, injective, as N_∞ is non-degenerate. Now

$$\operatorname{rank} N_\infty = \operatorname{rank} \sigma_\infty^* TS_{v_\infty} - \operatorname{rank} d\sigma_\infty(TM) = n - 2 = \operatorname{rank} S^\perp$$

shows that $\mathcal{Q}$ is an isometric isomorphism. Furthermore, for $\xi \in \Gamma(N_\infty)$,

$$\nabla^{S^\perp}(\mathcal{Q}(\xi)) = \pi_{S^\perp}(d\xi) + d(\xi, \mathcal{H}_\infty)\pi_{S^\perp}\sigma_\infty + (\xi, \mathcal{H}_\infty)\pi_{S^\perp}(d\sigma_\infty) = \pi_{S^\perp}(d\xi),$$

whilst

$$\mathcal{Q}(\nabla^{N_\infty}\xi) = \pi_{N_\infty}(d\xi) + (\pi_{N_\infty}(d\xi), \mathcal{H}_\infty)\sigma_\infty \in \Gamma(S^\perp).$$

To show that $\mathcal{Q}$ preserves connections, we just need to verify that $d\xi - \pi_{N_\infty}(d\xi) \in \Gamma(S)$. That is immediate: for $\xi \in \Gamma(N_\infty) \subset \Gamma(\langle \sigma_\infty, v_\infty \rangle^\perp)$, $d\xi$ is still a section of $\langle \sigma_\infty, v_\infty \rangle^\perp$,

$$(d\xi, \sigma_\infty) = (d\xi, \sigma_\infty) + (\xi, d\sigma_\infty) = 0 = (d\xi, v_\infty) + (\xi, dv_\infty) = (d\xi, v_\infty);$$

and, therefore,

$$d\xi - \pi_{N_\infty}(d\xi) = \pi_{d\sigma_\infty(TM)}(d\xi).$$

□

2.3. The Gauss-Ricci and Codazzi equations

2.3.1. The exterior power $\wedge^2\mathbb{R}^{n+1,1}$ et al.: a few utilities. This section consists of a collection of useful, well-known facts involving exterior products.

The space $\wedge^2\mathbb{R}^{n+1,1}$ can be identified with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$,

$$\wedge^2\, \mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1}), \tag{2.12}$$

via

$$w \mapsto (v_1 \wedge v_2)(w) := (w, v_1)v_2 - (w, v_2)v_1, \tag{2.13}$$

which assigns to $v_1 \wedge v_2$ a skew-symmetric transformation. We shall consider this identification throughout this work.

Under the identification (2.12) defined by (2.13), given a non-degenerate subbundle V of $\underline{\mathbb{R}}^{n+1,1}$, we have

$$\Gamma(V \wedge V^\perp) = \{\xi \in \Gamma(o(\underline{\mathbb{R}}^{n+1,1})) : \xi(V) \subset V^\perp, \xi(V^\perp) \subset V\},$$

as well as

$$\Gamma(\wedge^2 V \oplus \wedge^2 V^\perp) = \{\xi \in \Gamma(o(\underline{\mathbb{R}}^{n+1,1})) : \xi(V) \subset V, \xi(V^\perp) \subset V^\perp\},$$

and

$$o(\underline{\mathbb{R}}^{n+1,1}) = \wedge^2 V \oplus \wedge^2 V^\perp \oplus V \wedge V^\perp, \tag{2.14}$$

for the trivial bundle $o(\underline{\mathbb{R}}^{n+1,1}) := M \times o(\mathbb{R}^{n+1,1})$. It is immediate and very useful to observe that, given $\xi \in \Gamma(V \wedge V^\perp)$,

$$\xi_{|_{V^\perp}} = -(\xi_{|_V})^t, \tag{2.15}$$

for $(\xi_{|_V})^t$ the transpose of $\xi_{|_V} \in \Gamma(\mathrm{Hom}(V, V^\perp))$ with respect to the metric on $\underline{\mathbb{R}}^{n+1,1}$.

Given V and W subbundles of $\underline{\mathbb{R}}^{n+1,1}$, provided with connections ∇^V and ∇^W, respectively, the bundle $V \wedge W$ is provided with the metric induced from the one on

$\mathrm{End}(\underline{\mathbb{R}}^{n+1,1})$ and, canonically, with the connection given by

$$\nabla(v\wedge w):=(\nabla^V v)\wedge w+v\wedge(\nabla^W w),$$

for $v\in\Gamma(V), w\in\Gamma(W)$. In the particular case $V=\underline{\mathbb{R}}^{n+1,1}=W$ and $\nabla^V=\nabla^W$ is a metric connection, ∇ coincides with the connection canonically induced by ∇^V in $\mathrm{End}(\underline{\mathbb{R}}^{n+1,1})$, over sections of the trivial bundle $M\times\wedge^2\mathbb{R}^{n+1,1}=:\wedge^2\underline{\mathbb{R}}^{n+1,1}\cong o(\underline{\mathbb{R}}^{n+1,1})$. If ∇^V is metric and $V^\perp$ is provided with a connection, the correspondence

$$\eta\mapsto\eta|_V \tag{2.16}$$

defines a bundle isomorphism $V\wedge V^\perp\to\mathrm{Hom}(V,V^\perp)$ preserving metrics and connections, providing an identification

$$V\wedge V^\perp\cong\mathrm{Hom}(V,V^\perp)$$

of bundles provided with a metric and a connection. In fact, given $v,v^*\in\Gamma(V)$, $v^\perp,v^\perp_*\in\Gamma(V^\perp)$, $\eta=v\wedge v^\perp$ and $\eta^*=v^*\wedge v^\perp_*$,

$$\begin{aligned}(\nabla\eta|_V)v^* &= \nabla^{V^\perp}(\eta v^*)-\eta(\nabla^V v^*)\\ &= \nabla^{V^\perp}(v,v^*)v^\perp-(v,\nabla^V v^*)v^\perp\\ &= (v,v^*)\nabla^{V^\perp}v^\perp+d(v,v^*)v^\perp-(v,\nabla^V v^*)v^\perp\end{aligned}$$

and, therefore, as ∇^V is metric,

$$(\nabla\eta|_V)v^*=(v,v^*)\nabla^{V^\perp}v^\perp+(\nabla^V v,v^*)v^\perp=(\nabla\eta)v^*;$$

whilst, on the other hand, $(\eta^*|_V)^t\circ\eta|_{V^\perp}=(\eta*)^t|_{V^\perp}\circ\eta|_{V^\perp}=0$ and, therefore,

$$(\eta|_V,\eta^*|_V)=\mathrm{tr}((\eta^*|_V)^t\circ\eta|_V)=\mathrm{tr}((\eta^*)^t\circ\eta)=(\eta,\eta^*).$$

It will be useful to note that, as a straightforward computation shows, given $a,b\in\mathbb{R}^{n+1,1}$ and $T\in o(\mathbb{R}^{n+1,1})$,

$$[T,a\wedge b]=(Ta)\wedge b+a\wedge(Tb), \tag{2.17}$$

for the Lie bracket in $o(\mathbb{R}^{n+1,1})$; as well as, for $T\in O(\mathbb{R}^{n+1,1})$,

$$\mathrm{Ad}_T(a\wedge b)=Ta\wedge Tb. \tag{2.18}$$

Given a vector bundle P over M whose fibres are Lie algebras and $\mu,\eta\in\Omega^1(P)$, we define a 2-form $[\mu\wedge\eta]\in\Omega^2(P)$ by

$$[\mu\wedge\eta]_{(X,Y)}:=[\mu_X,\eta_Y]-[\mu_Y,\eta_X]$$

for $X,Y\in\Gamma(TM)$. Note that

$$[\eta\wedge\mu]=[\mu\wedge\eta]$$

and that

$$[\mu \wedge \mu]_{(X,Y)} = 2[\mu_X, \mu_Y].$$

In the case P is a bundle of endomorphisms, we define another 2-form with values in P, $\mu \wedge \eta \in \Omega^2(P)$, using composition of endomorphisms to multiply coefficients in the exterior product:

$$\mu \wedge \eta_{(X,Y)} := \mu_X \eta_Y - \mu_Y \eta_X,$$

for all X, Y. Note that, in that case,

$$[\mu \wedge \eta] = \mu \wedge \eta + \eta \wedge \mu. \tag{2.19}$$

Suppose M is provided with a conformal structure $\mathcal{C}$. Observe that

$$[\mu \wedge *\eta] = -[*\,\mu \wedge \eta]. \tag{2.20}$$

For this, fix a (locally) never-zero $Z \in \Gamma(T^{1,0}M)$ and verify that $[\mu \wedge *\eta](Z, \overline{Z}) = -[*\,\mu \wedge \eta](Z, \overline{Z})$, or, equivalently, that $-[\mu \wedge (\eta J)](Z, \overline{Z}) = [(\mu J) \wedge \eta](Z, \overline{Z})$, which is, in fact, an immediate consequence of the fact that Z and $\overline{Z}$ are eigenvectors of J associated to the eigenvalues i and $-i$, respectively. In particular,

$$[*\mu \wedge *\eta] = [\,\mu \wedge \eta]. \tag{2.21}$$

Note that, if μ and η are both either $(1,0)$-forms or $(0,1)$-forms, then $[\mu \wedge \eta]$ vanishes:

$$[\mu^{1,0} \wedge \eta^{1,0}] = 0 = [\mu^{0,1} \wedge \eta^{0,1}], \tag{2.22}$$

for all μ and η. Note also that, given $\xi_1, \xi_2 \in \Omega^1(o(\underline{\mathbb{R}}^{n+1,1}))$ and $T \in \mathrm{End}(\mathbb{R}^{n+1,1})$,

$$[\,\mathrm{Ad}_T\, \xi_1 \,\wedge\, \mathrm{Ad}_T\, \xi_2\,] = \mathrm{Ad}_T\, [\,\xi_1 \wedge \xi_2\,]. \tag{2.23}$$

Lastly, suppose that ∇^1 and ∇^2 are connections on $\underline{\mathbb{R}}^{n+1,1}$ related by

$$\nabla^1 = \nabla^2 + A,$$

for some $A \in \Omega^1(\mathrm{End}(\underline{\mathbb{R}}^{n+1,1}))$. The respective curvature tensors, R^{∇^1} and R^{∇^2}, are related by

$$R^{\nabla^1} = R^{\nabla^2} + d^{\nabla^2} A + \frac{1}{2}\,[A \wedge A], \tag{2.24}$$

whilst the corresponding exterior derivatives, d^{∇^1} and d^{∇^2}, relate by

$$d^{\nabla^1}\xi = d^{\nabla^2}\xi + [A \wedge \xi], \tag{2.25}$$

for $\xi \in \Omega^1(\mathrm{End}(\underline{\mathbb{R}}^{n+1,1}))$. As a final remark, note that, in the case ∇^2 is a metric connection, the connection ∇^1 is metric if and only if A is skew-symmetric.

2.3.2. The Gauss-Ricci and Codazzi equations. The central sphere congruence $S : M \to Gr_{(3,1)}(\mathbb{R}^{n+1,1})$ of a surface in n-space defines a decomposition $d = \mathcal{D} + \mathcal{N}$ of the trivial flat connection on $\underline{\mathbb{R}}^{n+1,1}$ into the sum of a connection $\mathcal{D}$, with respect

to which S and $S^\perp$ are parallel, and a 1-form $\mathcal{N}$ with values in $S\wedge S^\perp$. Explicitly, $\mathcal{D}:=\nabla^S+\nabla^{S^\perp}$, for ∇^S and $\nabla^{S^\perp}$ the connections on S and $S^\perp$, respectively, defined by orthogonal projection of d, and $\mathcal{N}:=d-\mathcal{D}$. The flatness of d encodes two structure equations on $\mathcal{D}$ and $\mathcal{N}$.

Define a connection $\mathcal{D}$ on $\underline{\mathbb{R}}^{n+1,1}$ by

$$\mathcal{D}:=\nabla^S\circ\pi_S+\nabla^{S^\perp}\circ\pi_{S^\perp}.$$

For simplicity, and only temporarily, denote π_S and $\pi_{S^\perp}$ by $(\,)^T$ and $(\,)^\perp$, respectively. Given $\mu\in\Gamma(\underline{\mathbb{R}}^{n+1,1})$,

$$d\mu=(d\mu^T)^T+(d\mu^T)^\perp+(d\mu^\perp)^T+(d\mu^\perp)^\perp=\mathcal{D}\mu+(d\mu^T)^\perp+(d\mu^\perp)^T$$

and, given $\eta\in\Gamma(\underline{\mathbb{R}}^{n+1,1})$,

$$((d\mu^T)^\perp,\eta)=(d\mu^T,\eta^\perp)=-(\mu^T,d\eta^\perp)=-(\mu,(d\eta^\perp)^T).$$

It is then clear that, for $\mu,\eta\in\Gamma(\underline{\mathbb{R}}^{n+1,1})$,

$$d(\mu,\eta)=(d\mu,\eta)+(\mu,d\eta)=(\mathcal{D}\mu,\eta)+(\mu,\mathcal{D}\eta);$$

$\mathcal{D}$ is a metric connection. Thus

$$d=\mathcal{D}+\mathcal{N} \tag{2.26}$$

defines a 1-form $\mathcal{N}\in\Omega^1(o(\underline{\mathbb{R}}^{n+1,1}))$. In fact,

$$\mathcal{N}=\pi_{S^\perp}\circ d\circ\pi_S+\pi_S\circ d\circ\pi_{S^\perp}\in\Omega^1(S\wedge S^\perp).$$

We may, alternatively, use, specifically, the notations $\mathcal{D}_S$ and $\mathcal{N}_S$ for, respectively, $\mathcal{D}$ and $\mathcal{N}$. It is very simple but very useful to note that

$$\mathcal{N}\Lambda=0. \tag{2.27}$$

Indeed, given $\sigma\in\Gamma(\Lambda)$ never-zero, $\mathcal{N}\sigma=\pi_{S^\perp}\circ d\sigma$ and $d\sigma\in\Omega^1(S)$. By the skew-symmetry of $\mathcal{N}$, it follows, in particular, that $\mathrm{Im}\,\mathcal{N}\subset\Lambda^\perp$ and, consequently, that $\mathcal{N}S^\perp\subset S\cap\Lambda^\perp=\Lambda^{(1)}$. Hence

$$\mathcal{N}\in\Omega^1(\Lambda^{(1)}\wedge S^\perp). \tag{2.28}$$

The flatness of d, characterized by

$$0=R^{\mathcal{D}}+d^{\mathcal{D}}\mathcal{N}+\frac{1}{2}[\mathcal{N}\wedge\mathcal{N}],$$

encodes two structure equations, as follows. The $\mathcal{D}$-parallelness of S and $S^\perp$ establishes $\mathcal{D}(\Gamma(S\wedge S^\perp))\subset\Omega^1(S\wedge S^\perp)$ and, therefore, $d^{\mathcal{D}}\mathcal{N}\in\Omega^2(S\wedge S^\perp)$. Together with the fact that $\mathcal{D}$ is metric, it establishes, on the other hand, $R^{\mathcal{D}}\in\Omega^2(\wedge^2S\oplus\wedge^2S^\perp)$. By equation

(2.17), we verify that $[\mathcal{N} \wedge \mathcal{N}] \in \Omega^2(\wedge^2 S \oplus \wedge^2 S^\perp)$. According to the decomposition (2.14), it follows then that:

Proposition 2.6. *(Gauss-Ricci equation)*

$$R^{\mathcal{D}} + \frac{1}{2}\,[\mathcal{N} \wedge \mathcal{N}] = 0;$$

and

Proposition 2.7. *(Codazzi equation)*

$$d^{\mathcal{D}}\mathcal{N} = 0.$$

CHAPTER 3

Surfaces under change of flat metric connection

In many occasions throughout this work, we use an interpretation of loop group theory by F. Burstall and D. Calderbank [**11**] and produce transformations of surfaces by the action of loops of flat metric connections. Specifically, by replacing the trivial flat connection by another flat metric connection $\tilde{d}$ on $\underline{\mathbb{R}}^{n+1,1}$, we transform (in certain cases) a surface $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ into a *$\tilde{d}$-surface* $\tilde{\Lambda}$, or, equivalently, into another surface $\tilde{\phi}\Lambda$, defined, up to a Möbius transformation,[1] for $\tilde{\phi} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ an isomorphism of bundles provided with a metric and a connection. Many will be the examples in this work of such transformations preserving the geometrical aspects of a class, i.e., establishing symmetries of integrable systems. This tiny chapter is merely introductory of the concept of $\tilde{d}$-surface.

Recall that the flatness of a bundle (over M) ensures the existence of a local frame (defined on a simply connected component of M) made up of parallel sections. Recall as well that, if the bundle is also provided with a metric with respect to which the connection is a metric connection, then there exists an orthonormal local frame constituted by parallel sections.

Let $\tilde{d}$ be a flat metric connection on $\underline{\mathbb{R}}^{n+1,1}$. Let $\mathcal{B} = (e_i)_i$ and $\tilde{\mathcal{B}} = (\tilde{e}_i)_i$ be orthonormal local frames of $\underline{\mathbb{R}}^{n+1,1}$, parallel with respect to d and $\tilde{d}$, respectively, $\tilde{d}\,\tilde{e}_i = 0 = d\,e_i$, for $i = 1, ..., n+2$; with e_{n+2} and $\tilde{e}_{n+2}$ time-like. Define an isometry $\phi_{\tilde{\mathcal{B}}\,\mathcal{B}} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ of bundles, preserving connections,

$$\phi_{\tilde{\mathcal{B}}\,\mathcal{B}} \circ \tilde{d} = d \circ \phi_{\tilde{\mathcal{B}}\,\mathcal{B}},$$

by setting $\phi_{\tilde{\mathcal{B}}\,\mathcal{B}}(\tilde{e}_i) = e_i$, for $i = 1, ..., n+2$. Observe that, although $\phi_{\tilde{\mathcal{B}}\,\mathcal{B}}$ depends on the choice of the frames $\mathcal{B}$ and $\tilde{\mathcal{B}}$, it is uniquely determined up to a Möbius transformation. For that, first suppose $\mathcal{B}' = (e'_i)_i$ is another d-parallel orthonormal local frame of $\underline{\mathbb{R}}^{n+1,1}$, with e'_{n+2} time-like, define $T \in \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$ by $T(e_i) = e'_i$, for all i, and note that $\phi_{\tilde{\mathcal{B}}\,\mathcal{B}'} = T\phi_{\tilde{\mathcal{B}}\,\mathcal{B}}$ and that T is constant:

$$T \circ d = \phi_{\tilde{\mathcal{B}}\,\mathcal{B}'}\phi_{\tilde{\mathcal{B}}\,\mathcal{B}}^{-1} \circ d = \phi_{\tilde{\mathcal{B}}\,\mathcal{B}'} \circ \tilde{d} \circ \phi_{\tilde{\mathcal{B}}\,\mathcal{B}}^{-1} = d \circ \phi_{\tilde{\mathcal{B}}\,\mathcal{B}'}\,\phi_{\tilde{\mathcal{B}}\,\mathcal{B}}^{-1} = d \circ T.$$

[1]At some point, we shall omit the indication "up to a Möbius transformation" and assume a Möbius geometry point of view.

A similar argument shows the independence, up to a Möbius transformation, of $\phi_{\tilde{\mathcal{B}}\mathcal{B}}$ with respect to the choice of the frame $\tilde{\mathcal{B}}$. Observe, on the other hand, that any isomorphism $\phi : (\underline{\mathbb{R}}^{n+1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1}, d)$ of bundles provided with a metric and a connection is of the form $\phi_{\tilde{\mathcal{B}}\mathcal{B}}$ for some $\mathcal{B}$ and $\tilde{\mathcal{B}}$: fixing an orthonormal $\tilde{d}$-parallel local frame $\tilde{\mathcal{B}} = (\tilde{e}_i)_i$ of $\underline{\mathbb{R}}^{n+1,1}$ and setting $\mathcal{B} := (\phi(\tilde{e}_i))_i$, we define an orthonormal d-parallel local frame of $\underline{\mathbb{R}}^{n+1,1}$ such that $\phi = \phi_{\tilde{\mathcal{B}}\mathcal{B}}$.

Throughout this text, given a vector bundle P, provided with a metric, and connections ∇ and ∇' on P, by isomorphism $(P, \nabla) \to (P, \nabla')$ shall be understood isomorphism of bundles provided with a metric and a connection.

Definition 3.1. *Given $V \subset \underline{\mathbb{R}}^{n+1,1}$ and an isomorphism*

$$\phi_{\tilde{d}} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d),$$

the bundle $\tilde{V} := \phi_{\tilde{d}} V$ is said to be the transformation of V defined, up to a Möbius transformation, by the flat metric connection $\tilde{d}$.

Henceforth, we shall omit the indication "up to a Möbius transformation" and assume a Möbius geometry point of view.

Remark 3.2. *Obviously, given ∇ and ∇' connections on $\underline{\mathbb{R}}^{n+1,1}$ and ϕ an endomorphism of $\underline{\mathbb{R}}^{n+1,1}$, the hypothesis $\phi \circ \nabla \circ \phi^{-1} = d = \phi \circ \nabla' \circ \phi^{-1}$ forces $\nabla = \nabla'$. In particular, given a flat metric connection $\hat{d} \neq \tilde{d}$ on $\underline{\mathbb{R}}^{n+1,1}$ and isomorphisms $\phi_{\hat{d}} : (\underline{\mathbb{R}}^{n+1,1}, \hat{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ and $\phi_{\tilde{d}} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$,*

$$\phi_{\hat{d}} \neq \phi_{\tilde{d}}.$$

Of course, this does not exclude the possibility of, given a subbundle V of $\underline{\mathbb{R}}^{n+1,1}$, the transformations of V defined by $\phi_{\tilde{d}}$ and $\phi_{\hat{d}}$ being the same.

Let us concentrate on the particular case $V = \Lambda$, a null line bundle, not necessarily defining an immersion into the projectivized light-cone; and on its transformation

$$\tilde{\Lambda} := \phi_{\tilde{d}} \Lambda,$$

into another null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$.

Definition 3.3. *We say that Λ is a $\tilde{d}$-surface if* $\mathrm{rank}\, \Lambda^{(1)}_{\tilde{d}} = 3$*, for*

$$\Lambda^{(1)}_{\tilde{d}} := \langle \sigma, \tilde{d}_{e_1} \sigma, \tilde{d}_{e_2} \sigma \rangle,$$

defined independently of the choices of a never-zero $\sigma \in \Gamma(\Lambda)$ and of a local frame $(e_i)_i$ of TM. In the particular case $\tilde{d} = d$, we shall, alternatively, omit the reference to $\tilde{d}$.

It is, perhaps, worth remarking that a $\tilde{d}$-surface is not necessarily a surface.

The fact that $\phi_{\tilde{d}}$ preserves the connections $\tilde{d}$ and d establishes

$$(\phi_{\tilde{d}} \Lambda)^{(1)} = \phi_{\tilde{d}} (\Lambda^{(1)}_{\tilde{d}}), \tag{3.1}$$

and, therefore, $\operatorname{rank}\tilde{\Lambda}^{(1)} = \operatorname{rank}\Lambda^{(1)}_{\tilde{d}}$, and, ultimately:

Proposition 3.4. *$\tilde{\Lambda}$ is a surface if and only if Λ is a $\tilde{d}$-surface.*

An alternative perspective on the transformation of Λ into $\phi_{\tilde{d}}\Lambda$, is, in this way, that of the transformation

$$\Lambda \subset (\underline{\mathbb{R}}^{n+1,1}, d) \mapsto \Lambda \subset (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}),$$

consisting of the change of the trivial flat connection on $\underline{\mathbb{R}}^{n+1,1}$ into the flat metric connection $\tilde{d}$.

In what little is left in this section, we introduce a few concepts on $\tilde{d}$-surfaces. Suppose Λ is a $\tilde{d}$-surface. In that case, given a never-zero section σ of Λ, we define a positive definite metric $g^{\tilde{d}}_{\sigma}$ by

$$g^{\tilde{d}}_{\sigma}(X,Y) := (\tilde{d}_X\sigma, \tilde{d}_Y\sigma),$$

for $X, Y \in \Gamma(TM)$. Indeed,

$$g^{\tilde{d}}_{\sigma} = g_{\phi_{\tilde{d}}\sigma} \in \mathcal{C}_{\tilde{\Lambda}} =: \mathcal{C}^{\tilde{d}}_{\Lambda}. \tag{3.2}$$

Definition 3.5. *We define the $\tilde{d}$*-central sphere congruence *of Λ by*

$$S^{\tilde{d}} := \langle \sigma, \tilde{d}_{e_1}\sigma, \tilde{d}_{e_2}\sigma, \sum_i \tilde{d}_{e_i}\tilde{d}_{e_i}\sigma\rangle = \phi_{\tilde{d}}^{-1}\, S_{\tilde{\Lambda}}, \tag{3.3}$$

independently of the choices of a never-zero $\sigma \in \Gamma(\Lambda)$ and of a local orthonormal frame $(e_i)_i$ of TM with respect to $g^{\tilde{d}}_{\sigma}$.

The non-degeneracy of $S_{\tilde{\Lambda}}$ ensures that of $S^{\tilde{d}}$. Let $\pi_{S^{\tilde{d}}}$ and $\pi_{(S^{\tilde{d}})^{\perp}}$ be the orthogonal projections of

$$\underline{\mathbb{R}}^{n+1,1} = S^{\tilde{d}} \oplus (S^{\tilde{d}})^{\perp}$$

onto $S^{\tilde{d}}$ and $(S^{\tilde{d}})^{\perp}$, respectively. We define a connection $\mathcal{D}^{\tilde{d}}$ on $\underline{\mathbb{R}}^{n+1,1}$ by

$$\mathcal{D}^{\tilde{d}} := \pi_{S^{\tilde{d}}} \circ \tilde{d} \circ \pi_{S^{\tilde{d}}} + \pi_{(S^{\tilde{d}})^{\perp}} \circ \tilde{d} \circ \pi_{(S^{\tilde{d}})^{\perp}}$$

and a 1-form $\mathcal{N}^{\tilde{d}} \in \Omega^1(\operatorname{End}(\underline{\mathbb{R}}^{n+1,1}))$ by

$$\mathcal{N}^{\tilde{d}} := \tilde{d} - \mathcal{D}^{\tilde{d}}.$$

Note that

$$\mathcal{D}^{\tilde{d}} = \phi_{\tilde{d}}^{-1} \circ \mathcal{D}_{\tilde{\Lambda}} \circ \phi_{\tilde{d}}, \quad \mathcal{N}^{\tilde{d}} = \phi_{\tilde{d}}^{-1}\mathcal{N}_{\tilde{\Lambda}}\phi_{\tilde{d}}. \tag{3.4}$$

CHAPTER 4

Willmore surfaces

Among the classes of Riemannian submanifolds, there is that of Willmore surfaces, named after T. Willmore [**60**] (1965), although the topic was mentioned by W. Blaschke [**4**] (1929) and by G. Thomsen [**55**] (1923). Early in the nineteenth century, S. Germain [**28**], [**29**] studied elastic surfaces. On her pioneering analysis, she claimed that the elastic force of a thin plate is proportional to its mean curvature. Since then, the mean curvature remains a key concept in theory of elasticity. In modern literature on the elasticity of membranes (see, for example, [**37**] and [**40**]), a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature is considered the elastic energy of a membrane. By neglecting the total mean curvature (by physical considerations) and having in consideration that the total Gaussian curvature of compact orientable Riemannian surfaces without boundary is a topological invariant, T. Willmore defined the Willmore energy of a compact oriented Riemannian surface, without boundary, isometrically immersed in $\mathbb{R}^3$, to be $\mathcal{W} = \int H^2 dA$. The Willmore functional "extends" to isometric immersions of compact oriented Riemannian surfaces in Riemannian manifolds by means of half of the total squared norm of the trace-free part of the second fundamental form, which, in fact, amongst surfaces in $\mathbb{R}^3$, differs from $\mathcal{W}$ by the total Gaussian curvature, but still shares then the critical points with $\mathcal{W}$. Willmore surfaces are the extremals of the Willmore functional. W. Blaschke [**4**] established the Möbius invariance of the Willmore energy of a surface in spherical 3-space. B.-Y. Chen [**18**] generalized it to surfaces in constant curvature Riemnannian manifolds. We present a manifestly conformally invariant formulation of the Willmore energy of a surface in n-dimensional space-form, $\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (\mathcal{N} \wedge *\mathcal{N})$. The class of Willmore surfaces in n-space is then established as invariant under the group of Möbius transformations of S^n.[1] As already known to Blaschke [**4**] for the particular case of spherical 3-space, the Willmore energy of a surface in a space-form coincides with the energy of its central sphere congruence. Furthermore, a result by Blaschke [**4**] (for $n = 3$) and N. Ejiri [**27**] (for general n) characterizes Willmore surfaces in spherical n-space by the harmonicity of the central sphere congruence. Via this characterization, the class of Willmore surfaces in space-forms is then associated to a class of harmonic

[1] In fact, we verify the Möbius invariance of the Willmore energy of a general surface and establish then the Möbius invariance of the class of Willmore surfaces.

maps into Grassmannians. This enables us to apply to this class of surfaces the well-developed integrable systems theory of harmonic maps into Grassmannian manifolds, with a spectral deformation and Bäcklund transformations, cf. [**54**] and [**56**]. We define in this way a spectral deformation of Willmore surfaces, which we verify to coincide, up to reparametrization, with the one presented in [**14**], as well as *Bäcklund transformations*, the latter arising from a more complex construction, presented in a chapter below.

4.1. The Willmore functional

In this section, we present a manifestly conformally invariant formulation of the Willmore energy of a surface in a space-form.

We start by recalling the classical concept of Willmore energy of an isometric immersion of a Riemannian surface into a Riemannian manifold.

Consider a Riemannian manifold $(\bar{M}, g)$ and an immersion $f : M \to \bar{M}$. Provide M with the metric g_f induced by f from g, making f into an isometric immersion. Recall the trace-free part of the second fundamental form of f,

$$\Pi^0 = \Pi - g_f \otimes \mathcal{H} \in \Gamma(L^2(TM, N_f)).$$

Suppose M is compact. The Willmore energy of f is defined to be[2]

$$\mathcal{W}(f) := \int_M |\Pi^0|^2 \mathrm{d}A,$$

for

$$|\Pi^0|^2 := \sum_{i,j} (\Pi^0(X_i, X_j), \Pi^0(X_i, X_j)),$$

defined independently of the choice of a local orthonormal frame $(X_i)_{i=1,2}$ of TM, and $\mathrm{d}A$ the area element of M. Let g' be a metric on $\bar{M}$ conformally equivalent to g, $g' = e^{2u} g$, for some $u \in C^\infty(\bar{M}, \mathbb{R})$. Let g'_f denote the metric induced in M by f from g' and Π' denote the second fundamental form of $f : M \to (\bar{M}, g')$. Following (2.7), we conclude that the trace-free part of the second fundamental form is invariant under conformal changes of the metric,

$$\Pi^0 = (\Pi^0)',$$

[2]In fact, the Willmore energy is classically defined as *half* of the total squared norm of the trace-free part of the second fundamental form. In either case, it generalizes the Willmore energy of a surface in $\mathbb{R}^3$, as defined by T. Willmore, only up to some constant and, in this case, some scaling (as we shall verify later on). Although not sharing extremes, all these different energies share extremals. The reason for this scaling of the Willmore energy by 2 is avoiding some scaling when comparing the Willmore energy of a surface to the energy of its central sphere congruence, to take place in section 4.3 below.

for Π^0 and $(\Pi^0)'$ the trace-free parts of Π and Π', respectively. Ultimately, we conclude that

$$(|(\Pi^0)'|')^2 = e^{-2u\circ f}\,|\Pi^0|^2,$$

with $'$ indicating, yet again, "with respect to g' ". On the other hand, according to Lemma 1.6,

$$dA' = e^{2u\circ f}dA,$$

relating the area element dA' of (M, g'_f) to the area element of (M, g_f). Under a conformal change of the metric, the square of the length of Π^0 and the area element of M change in an inverse way, leaving the Willmore energy unchanged. It follows that:

Theorem 4.1. *The Willmore energy is a Möbius invariant.*

The Möbius invariance of the Willmore energy of a surface in spherical 3-space was first established by W. Blaschke [**4**]. B.-Y. Chen [**18**] generalized it to surfaces in constant curvature Riemannian manifolds[3].

Next we present a manifestly conformally invariant formulation of the Willmore energy of a surface in a space-form. Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone. We consider $o(\underline{\mathbb{R}}^{n+1,1})$ provided with the metric induced by $\mathrm{End}(\underline{\mathbb{R}}^{n+1,1})$: given $\alpha, \beta \in o(\mathbb{R}^{n+1,1})$, $(\alpha, \beta) = -\mathrm{tr}\,\alpha\beta$. Fixing a conformal structure in M, and having in consideration the invariance of the Hodge $*$-operator on 1-forms over M under conformal changes of the metric on M, we have well-defined a 2-form $(\mathcal{N} \wedge *\mathcal{N})$ over M with values in $\mathbb{R}$.

Definition 4.2. *We define the* Willmore energy *of* Λ *to be*

$$\mathcal{W}(\Lambda) := \frac{1}{2}\int_M (\mathcal{N} \wedge *\mathcal{N}),$$

with respect to the conformal structure induced in M *by* Λ.

The previous definition follows the definition of energy of the mean curvature sphere congruence[4] of a surface in spherical 4-space, presented in [**12**]. Now fix a non-zero v_∞ in $\mathbb{R}^{n+1,1}$ and consider the surface $\sigma_\infty : M \to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. The immersions Λ and σ_∞ are related by

$$\Lambda = \pi_{\mathcal{L}}|_{S_{v_\infty}} \circ \sigma_\infty$$

via the conformal diffeomorphism $\pi_{\mathcal{L}}|_{S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L})\backslash\mathbb{P}(\mathcal{L}\cap\langle v_\infty\rangle^\perp)$. Hence the Willmore energy of $\Lambda : M \to (\mathbb{P}(\mathcal{L}), h)$, fixing $h \in \mathcal{C}_{\mathbb{P}(\mathcal{L})}$, coincides with the Willmore energy of σ_∞. The Willmore energy of the conformal immersion $\Lambda : (M, \mathcal{C}_\Lambda) \to (\mathbb{P}(\mathcal{L}), \mathcal{C}_{\mathbb{P}(\mathcal{L})})$ consists of the Willmore energy of Λ as an immersion of M into the projectivized light-cone provided with a metric in $\mathcal{C}_{\mathbb{P}(\mathcal{L})}$ (chosen arbitrarily), as established next:

[3]We shall compute the Willmore energy in this special case later on in this section.

[4]For the relationship between the mean curvature sphere congruence and the central sphere congruence, see Section 9.1.3.

Theorem 4.3. *The Willmore energy of the conformal immersion Λ coincides with the Willmore energy of σ_∞,*

$$\mathcal{W}(\Lambda) = \mathcal{W}(\sigma_\infty). \tag{4.1}$$

Proof. The Willmore energy of σ_∞ is given by $\int_M |\Pi^0_\infty|^2 \mathrm{d}A_\infty$, for Π^0_∞ the trace-free part of Π_∞ and $\mathrm{d}A_\infty$ the area element of M when provided with the metric g_∞. On the other hand, $(\mathcal{N} \wedge *\mathcal{N}) = -(*\mathcal{N} \wedge \mathcal{N})$ is a conformally invariant way of writing $(\mathcal{N}, \mathcal{N})_g dA_g$,

$$(\mathcal{N} \wedge *\mathcal{N}) = (\mathcal{N}, \mathcal{N})_g dA_g,$$

for g in $\mathcal{C}_\Lambda$, with dA_g denoting the area element of (M, g) and $(\,,\,)_g$ denoting the Hilbert-Schmidt metric on $L((TM, g), o(\underline{\mathbb{R}}^{n+1,1}))$. In particular, $(\mathcal{N} \wedge *\mathcal{N}) = (\mathcal{N}, \mathcal{N})_{g_\infty} dA_\infty$. The proof of the theorem will consist of showing that $(\mathcal{N}, \mathcal{N})_{g_\infty} = 2|\Pi^0_\infty|^2$.

For simplicity, set $\alpha := \pi_{S^\perp} \circ d \circ \pi_S \in \Omega^1(\mathrm{End}(\underline{\mathbb{R}}^{n+1,1}))$. According to equation (2.15),

$$\mathcal{N} = \mathcal{N}|_S + \mathcal{N}|_{S^\perp} = \alpha - \alpha^t,$$

where t indicates the transpose with respect to the metric on $\mathbb{R}^{n+1,1}$. Fix a local orthonormal frame $\{X_i\}_i$ of TM with respect to g_∞. Obviously, $\alpha\alpha = 0$, so

$$(\mathcal{N}, \mathcal{N})_{g_\infty} = \sum_i (\mathcal{N}_{X_i}, \mathcal{N}_{X_i}) = -\sum_i \mathrm{tr}(\mathcal{N}_{X_i}\mathcal{N}_{X_i}) = \sum_i (\mathrm{tr}(\alpha_{X_i}\alpha^t_{X_i}) + \mathrm{tr}(\alpha^t_{X_i}\alpha_{X_i}))$$

and, therefore,

$$(\mathcal{N}, \mathcal{N})_{g_\infty} = 2\sum_i \mathrm{tr}(\alpha^t_{X_i}\alpha_{X_i}|_S),$$

having in consideration that $\alpha^t_{X_i}\alpha_{X_i}$ vanishes on $S^\perp$. Recall that, if $(e_i)_i$ and $(\hat{e}_i)_i$ are dual basis of a vector space E provided with a metric $(\,,\,)$ (possibly with signature), i.e., bases related by $(e_i, \hat{e}_j) = \delta_{ij}, \forall i, j$, then, given $\mu \in \mathrm{End}(E)$, $\mathrm{tr}(\mu) = \sum_i (\mu(e_i), \hat{e}_i)$. We shall now get two dual frames of S, in order to compute $\mathrm{tr}(\alpha^t_{X_i}\alpha_{X_i}|_S)$. Observe that, together, the conditions $(\hat{\sigma}_\infty, \hat{\sigma}_\infty) = 0$, $(\sigma_\infty, \hat{\sigma}_\infty) = -1$ and $(\hat{\sigma}_\infty, d\sigma_\infty) = 0$ determine uniquely a section $\hat{\sigma}_\infty$ of S. In fact, as $d\sigma_\infty(TM)$ is a bundle of $(2,0)$-planes, its orthogonal complement in S is a bundle of $(1,1)$-planes,

$$(d\sigma_\infty(TM))^\perp \cap S = \underline{\mathbb{R}}^{1,1},$$

which, by the nullity of $\hat{\sigma}_\infty$, restricts $\hat{\sigma}_\infty$ to two light lines, one of which is $\langle \sigma_\infty \rangle$. The condition $(\sigma_\infty, \hat{\sigma}_\infty) = -1$ shows that $\hat{\sigma}_\infty$ is not in $\langle \sigma_\infty \rangle$ and, ultimately, determines $\hat{\sigma}_\infty$. The metric relation between σ_∞ and $\hat{\sigma}_\infty$ shows that $\hat{\sigma}_\infty \notin \Lambda^{(1)}$, telling us that $(\sigma_\infty, d_{X_1}\sigma_\infty, d_{X_2}\sigma_\infty, \hat{\sigma}_\infty)$ forms a frame of S. The dual frame of S is the frame $(-\hat{\sigma}_\infty, d_{X_1}\sigma_\infty, d_{X_2}\sigma_\infty, -\sigma_\infty)$. Thus

$$\begin{aligned} \mathrm{tr}(\alpha^t_{X_i}\alpha_{X_i}|_S) &= (\alpha^t_{X_i}\alpha_{X_i}(\sigma_\infty), -\hat{\sigma}_\infty) + (\alpha_{X_i}(d_{X_1}\sigma_\infty), \alpha_{X_i}(d_{X_1}\sigma_\infty)) \\ &\quad + (\alpha_{X_i}(d_{X_2}\sigma_\infty), \alpha_{X_i}(d_{X_2}\sigma_\infty)) + (\alpha_{X_i}(\hat{\sigma}_\infty), -\alpha_{X_i}(\sigma_\infty)) \end{aligned}$$

and, consequently,

$$\begin{aligned}\mathrm{tr}(\alpha^t_{X_i}\alpha_{X_i}|_S) &= \sum_j(\alpha_{X_i}(d_{X_j}\sigma_\infty),\alpha_{X_i}(d_{X_j}\sigma_\infty))\\ &= \sum_j(\mathcal{N}_{X_i}(d_{X_j}\sigma_\infty),\mathcal{N}_{X_i}(d_{X_j}\sigma_\infty)).\end{aligned}$$

Now recall the identification between N_∞ and $S^\perp$ via the isometric isomorphism $\mathcal{Q}$, presented in Proposition 2.5. For $\xi\in N_\infty\subset\langle\sigma_\infty,v_\infty\rangle^\perp$,

$$(\mathcal{Q}(\xi),\pi_{S^\perp}(v_\infty))=(\mathcal{Q}(\xi),v_\infty)=(\xi,v_\infty)+(\xi,\mathcal{H}_\infty)(\sigma_\infty,v_\infty)=-(\mathcal{Q}(\xi),\mathcal{Q}(\mathcal{H}_\infty)),$$

establishing $(\mu,\pi_{S^\perp}(v_\infty)+\mathcal{Q}(\mathcal{H}_\infty))=0$, for all $\mu\in S^\perp$, and, therefore,

$$\pi_{S^\perp}(v_\infty)=-\mathcal{Q}(\mathcal{H}_\infty). \tag{4.2}$$

Let π_{N_∞} denote the orthogonal projection of $\underline{\mathbb{R}}^{n+1,1}=d\sigma_\infty(TM)\oplus N_\infty\oplus\langle v_\infty,\sigma_\infty\rangle$ onto N_∞. For arbitrary $X,Y\in\Gamma(TM)$, write $d_Xd_Y\sigma_\infty=\gamma+\eta+\beta v_\infty+\lambda\sigma_\infty$, with $\gamma\in\Gamma(d\sigma_\infty(TM)),\eta\in\Gamma(N_\infty)$ and $\beta,\lambda\in\Gamma(\underline{\mathbb{R}})$. In fact, we can be more precise:

$$\beta=-(d_Xd_Y\sigma_\infty,\sigma_\infty)=(d_Y\sigma_\infty,d_X\sigma_\infty)=g_\infty(X,Y).$$

By equation (4.2), $\pi_{S^\perp}(d_Xd_Y\sigma_\infty)=\pi_{S^\perp}(\eta)-g_\infty(X,Y)\mathcal{Q}(\mathcal{H}_\infty)$. On the other hand, $\eta-\mathcal{Q}(\eta)=(\eta,\mathcal{H}_\infty)\sigma_\infty\in\Gamma(S)$ and, therefore, $\pi_{S^\perp}(\eta)=\mathcal{Q}\left(\pi_{N_\infty}(d_Xd_Y\sigma_\infty)\right)$. Thus $\pi_{S^\perp}(d_Xd_Y\sigma_\infty)=\mathcal{Q}(\pi_{N_\infty}(d_Xd_Y\sigma_\infty)-g_\infty(X,Y)\mathcal{H}_\infty)$, or, equivalently,

$$\mathcal{N}_X(d_Y\sigma_\infty)=\mathcal{Q}(\Pi^0_\infty(X,Y)).$$

It follows that

$$|\Pi^0_\infty|^2=\sum_{i,j}(\mathcal{N}_{X_i}(d_{X_j}\sigma_\infty),\mathcal{N}_{X_i}(d_{X_j}\sigma_\infty)), \tag{4.3}$$

completing the proof. □

We complete this section by computing the Willmore energy of a compact surface immersed in $\mathbb{R}^3$, specially popular in the literature. Consider an immersion f of M into $\mathbb{R}^3$ and provide M with the metric g_f induced by f. Fixing an orthonormal frame X_1,X_2 of TM, we have

$$\begin{aligned}|\Pi^0|^2 &= \sum_{i,j}(\Pi(X_i,X_j)-\delta_{ij}\mathcal{H},\Pi(X_i,X_j)-\delta_{ij}\mathcal{H})\\ &= \sum_{i,j}(\Pi(X_i,X_j),\Pi(X_i,X_j))-2(\sum_i\Pi(X_i,X_i),\mathcal{H})+2|\mathcal{H}|^2\\ &= \sum_{i,j}(\Pi(X_i,X_j),\Pi(X_i,X_j))-2|\mathcal{H}|^2,\end{aligned}$$

and, consequently,

$$\begin{aligned}|\Pi^0|^2 - 2|\mathcal{H}|^2 &= \sum_{i,j}(\Pi(X_i, X_j), \Pi(X_i, X_j)) - \sum_{i,j}(\Pi(X_i, X_i), \Pi(X_j, X_j)) \\ &= \sum_{i\neq j}(\Pi(X_i, X_j), \Pi(X_i, X_j)) - \sum_{i\neq j}(\Pi(X_i, X_i), \Pi(X_j, X_j)) \\ &= 2(\Pi(X_1, X_2), \Pi(X_1, X_2)) - 2(\Pi(X_1, X_1), \Pi(X_2, X_2)).\end{aligned}$$

Hence, by equation (2.5),

$$|\Pi^0|^2 = 2(|\mathcal{H}|^2 - K + \bar{K})$$

and, therefore,

$$\mathcal{W}(f) = 2\int_M (|\mathcal{H}|^2 - K + \bar{K})\,\mathrm{d}A.$$

In the particular case $\bar{M} = \mathbb{R}^3$, we get twice as much the famous Willmore energy of a compact surface immersed in $\mathbb{R}^3$:

$$\mathcal{W}(f) = 2\int_M (H^2 - K)\,\mathrm{d}A,$$

where H^2 denotes the square of the mean curvature of f (with respect to either of the two unit normal vector fields to f). Amongst compact surfaces without boundary in $\mathbb{R}^3$, and since for these the total Gaussian curvature is a topological invariant (cf. Gauss-Bonnet theorem), the Willmore functional shares critical points with the functional $\tilde{\mathcal{W}}$ given by

$$\tilde{\mathcal{W}}(f) := \int_M H^2\,\mathrm{d}A,$$

which is what T. Willmore [**60**] defined as the Willmore energy of a compact surface, without boundary, immersed in $\mathbb{R}^3$.

4.2. Willmore surfaces: definition and examples

Willmore surfaces are the critical points of the Willmore functional. In view of the Möbius invariance of the Willmore energy, Willmore surfaces form a Möbius invariant class of surfaces. Minimal surfaces in 3-dimensional space-forms, and so their Möbius transforms, are examples of Willmore surfaces.

Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone. Suppose M is compact.

Definition 4.4. *Λ is said to be a* Willmore surface *if*

$$\frac{d}{dt}_{|t=0}\mathcal{W}(\Lambda_t) = 0$$

for every variation $(\Lambda_t)_t$ of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$.

Classically, a Willmore surface is defined to be an immersion $f : M \to \bar{M}$ of M into a Riemannian manifold $\bar{M}$, for which $\frac{d}{dt}_{|t=0}\mathcal{W}(f_t) = 0$, for every variation $(f_t)_t$ of f through immersions $f_t : M \to \bar{M}$. It is immediate from Theorem 4.1 that conformal diffeomorphisms transform Willmore surfaces into Willmore surfaces.

Theorem 4.5. *The class of Willmore surfaces is Möbius invariant.*

In particular:

Proposition 4.6. Λ *is a Willmore surface if and only if, fixing* $v_\infty \in \mathbb{R}^{n+1,1}$ *non-zero, so is the surface in* S_{v_∞} *defined by* Λ.

In Section 4.6, we extend the concept of Willmore surface to surfaces that are, in particular, not necessarily compact. We verify that minimal surfaces in space-forms, and so their Möbius transforms, are examples of Willmore surfaces (see Section 8.2). In particular, the stereographic projection of a minimal surface in S^n is a Willmore surface in $\mathbb{R}^n$. The Clifford torus is embedded in S^3 as a minimal surface. It projects stereographically onto the $\sqrt{2}$ anchor ring, which is then a Willmore surface in $\mathbb{R}^3$ (as well as its Möbius transforms).

B. Lawson [**39**] proved that there are minimal embeddings into S^3 of surfaces of arbitrary genus. Thus there exist in $\mathbb{R}^3$ Willmore surfaces of arbitrary genus. But are all Willmore surfaces in $\mathbb{R}^3$ obtainable as stereographics projections of minimal surfaces in S^3? The answer is "no". In fact, J. Langer and D. Singer [**38**] showed that there are infinitely many closed curves on S^2 whose corresponding *Hopf torus* is a Willmore surface in S^3. And U. Pinkall [**50**] showed then that, with one exception, the Willmore surfaces obtained by stereographic projection of these Willmore tori in S^3 cannot possibly be obtained by stereographic projection of minimal surfaces in S^3. The exception is the $\sqrt{2}$ anchor ring.

4.3. Willmore energy vs. energy of the central sphere congruence

Under the standard identification $S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp$, of bundles provided with a metric, $dS = \mathcal{N}$, which establishes the Willmore energy of a surface conformally immersed in a space-form as the energy of its central sphere congruence.

Recall that, given P a compact oriented Riemannian manifold and Q a pseudo-Riemannian manifold, the energy of a smooth map $\phi : P \to Q$ is defined as

$$E(\phi) = \frac{1}{2}\int_P |d\phi|^2 \mathrm{dvol}_P,$$

for the Hilbert-Schmidt norm $|d\phi|$ of $d\phi \in \Gamma(\mathrm{Hom}(TP, \phi^*TQ))$ induced by the pseudo-Riemannian structures of P and Q, and dvol_P the volume element of P. If we think of the map ϕ as a way to confine and stretch an elastic P inside Q, then $E(\phi)$ represents

an elastic deformation energy. As observed, in particular, by J. Eells and J. Sampson [**26**]:

Lemma 4.7. *In the case P is 2-dimensional, the energy of $\phi : P \to Q$ can be invariantly defined with respect to a conformal class of metrics in P.*

Proof. If P is 2-dimensional, then under conformal changes of the metric on P, the volume element and the square norm of $d\phi$ vary in an inverse way. In fact, given g and $g' := e^u g$, for some $u \in \Gamma(\underline{\mathbb{R}})$, conformally equivalent metrics on P, we have, according to Lemma 1.6,

$$\mathrm{dvol}_{(P,g')} = e^u \mathrm{dvol}_{(P,g)},$$

whereas, clearly,

$$|d\phi|^2_{g'} = \frac{1}{e^u}\,|d\phi|^2_g.$$

□

We are then, in the case M is compact, in a position to discuss the energy $E(S, \mathcal{C}_\Lambda)$ of the central sphere congruence $S : M \to \mathcal{G}$ of a surface $\Lambda : M \to \mathbb{P}(\mathcal{L})$, when providing M with the conformal structure $\mathcal{C}_\Lambda$ induced in M by Λ. As already known to Blaschke [**4**] in the particular case of spherical 3-space:

Theorem 4.8. *The Willmore energy of a surface $\Lambda : M \to \mathbb{P}(\mathcal{L})$ conformally immersed in the projectivized light-cone coincides with the energy of its central sphere congruence $S : M \to \mathcal{G}$,*

$$\mathcal{W}(\Lambda) = E(S, \mathcal{C}_\Lambda).$$

The proof of the theorem will be immediate after a few considerations, as follows.

Under the identification $S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp)$, of bundles provided with a metric, defined in Section 2.1, we have $(d_X S)\xi = \pi_{S^\perp}(d_X \xi) = \mathcal{N}_X \xi$, for $\xi \in \Gamma(S)$, and, therefore, $d_X S = \mathcal{N}_X|_S$, given $X \in \Gamma(TM)$. Under, furthermore, the identification $\mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp$, of bundles provided with a metric, defined in Section 2.3.1, we have $d_X S = \mathcal{N}_X$, for all $X \in \Gamma(TM)$. Hence, under the identification

$$S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp,$$

of bundles provided with a metric, we have

$$dS = \mathcal{N} \tag{4.4}$$

and, therefore, fixing a metric on M,

$$|dS|^2 = |\mathcal{N}|^2.$$

The proof of Theorem 4.8 is now immediate:

Proof. Fixing a metric in $\mathcal{C}_\Lambda$,

$$(\mathcal{N} \wedge *\mathcal{N}) = -(*\mathcal{N} \wedge \mathcal{N}) = (\mathcal{N}, \mathcal{N})\, dA = |dS|^2 dA.$$

$\square$

4.4. Willmore surfaces and harmonicity

Willmore surfaces are the extremals of the Willmore functional, just like harmonic maps are the extremals of the energy functional. The Willmore energy of a surface conformally immersed in a space-form coincides with the energy of its central sphere congruence. Furthermore, a result by Blaschke [**4**] (for $n = 3$) and N. Ejiri [**27**] (for general n) characterizes Willmore surfaces isometrically immersed in spherical n-space by the harmonicity of the central sphere congruence. This characterization will enable us, in the sections below, to apply to the class of Willmore surfaces conformally immersed in space-forms the well-developed integrable systems theory of harmonic maps into Grassmanian manifolds and to prove that Willmore surfaces constitute an integrable system.

Recall that, given P a compact oriented Riemannian manifold and Q a pseudo-Riemannian manifold, a smooth map $\phi : P \to Q$ is said to be harmonic if it extremises the energy functional,

$$\frac{d}{dt}_{|t=0} E(\phi_t) = 0,$$

for every variation $(\phi_t)_t$ of ϕ through smooth maps from P to Q. The associated Euler-Lagrange equation is

$$\operatorname{tr} \nabla d\phi = 0, \tag{4.5}$$

where $\nabla d\phi$ denotes the Hessian of ϕ, the section of $S^2(TP, \phi^*TQ)$ defined by

$$\nabla d\phi(X, Y) := \nabla^{\phi^*TQ}_X d\phi(Y) - d\phi(\nabla^{TP}_X Y),$$

for the Levi-Civita connection ∇^{TP} on P and the connection ∇^{ϕ^*TQ} induced in the pull-back bundle ϕ^*TQ by the Levi-Civita connection on Q. The section $\tau_\phi := \operatorname{tr} \nabla d\phi$ of $\phi^{-1}TQ$ is called the tension field of ϕ. Behind the characterization of the harmonicity of ϕ provided by equation (4.5) is the classical formula

$$\frac{d}{dt}_{|t=0} E(\phi_t) = -\int_P (\dot{\phi}, \operatorname{tr}\nabla d\phi) \mathrm{dvol}_P, \tag{4.6}$$

relating the variation of energy through a variation of ϕ to the respective variational vector field,

$$\dot{\phi} := \frac{d}{dt}_{|t=0} \phi_t \in \Gamma(\phi^*TQ).$$

Equation (4.6) will be useful in the future. It will also be useful to recall that, as we travel along all the variations of ϕ, the variational vector field $\dot{\phi}$ travels along all the sections of ϕ^*TQ: given $\eta \in \Gamma(\phi^*TQ)$, by setting

$$\phi_t(p) := \exp^Q_{\phi(p)}(t\eta_p)$$

for $t \in \mathbb{R}, p \in P$, we define a variation $(\phi_t)_t$ of ϕ through smooth maps from P to Q for which $\dot{\phi} = \eta$.

Although the Levi-Civita connection is not a conformal invariant, the harmonicity of a map defined on a 2-dimensional manifold is preserved by conformal changes of the metric on that manifold. In fact, as observed by J. Eells and J. Sampson [**26**], energy and, therefore, harmonicity of a map of a surface are preserved by conformal diffeomorphisms. It is well known that

$$d^\nabla * d\phi = - * (\mathrm{tr}\nabla d\phi), \tag{4.7}$$

denoting ∇^{ϕ^*TQ} by ∇, which leads us to the following:

Lemma 4.9. *In the case P is 2-dimensional, the equation*

$$d^\nabla * d\phi = 0 \tag{4.8}$$

constitutes a characterization of the harmonicity of ϕ, manifestly invariant under a conformal change of the metric on P.

Proof. Equation (4.8) provides a characterization of the harmonicity of ϕ, according to equation (4.7). On the other hand, in the case P is 2-dimensional, the Hodge $*$-operator on 1-forms over P is a conformal invariant. Since d^∇ depends only on the pseudo-Riemannian structure on Q, we conclude that, under a conformal change of the metric on P, $d^\nabla * d\phi$ remains invariant (and so does then the harmonicity of ϕ). $\square$

There is then no ambiguity in the following statement. Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone. Suppose M is compact.

Theorem 4.10. *Λ is a Willmore surface if and only if its central sphere congruence $S : M \to \mathcal{G}$ is harmonic with respect to the conformal structure induced in M by Λ.*

The characterization of Willmore surfaces isometrically immersed in spherical n-space by the harmonicity of the central sphere congruence is due to W. Blaschke [**4**] (for $n = 3$) and N. Ejiri [**27**] (for general n). The proof of Theorem 4.10 we present next is a generalization in the light-cone picture of the proof presented in [**12**], in the quaternionic setting, for the particular case of surfaces in S^4. The conclusion will follow easily from three useful lemmas we present next.

Lemma 4.11. *Let $(\Lambda_t)_t$ be a variation of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$ and $(S_t)_t$ be the corresponding variation of S through central sphere congruences. Then*

$$\frac{d}{dt}_{|t=0}\mathcal{W}(\Lambda_t) = \frac{d}{dt}_{|t=0}E(S_t, \mathcal{C}_\Lambda), \tag{4.9}$$

for $E(S_t, \mathcal{C}_\Lambda)$ the energy of $S_t : M \to \mathcal{G}$ when providing M with the conformal structure $\mathcal{C}_\Lambda$.

Proof. Write $\mathcal{C}_t$ for the conformal structure induced in M by Λ_t, writing also $\mathcal{C}_\Lambda$ for $\mathcal{C}_0$. According to Theorem 4.8, for each t, $\mathcal{W}(\Lambda_t) = E(S_t, \mathcal{C}_t)$. The proof will consist of showing that

$$\frac{d}{dt}_{|t=0}E(S_t, \mathcal{C}_t) = \frac{d}{dt}_{|t=0}E(S_t, \mathcal{C}_\Lambda).$$

For each t,

$$E(S_t, \mathcal{C}_t) = \int_M \frac{1}{2}\,(dS_t, dS_t)_t \mathrm{dA}_t = \int_M \frac{1}{2}\,(dS_t \wedge *_t dS_t),$$

for $(,)_t$ the Hilbert-Schmidt metric on $\mathrm{Hom}((TM, g_t), \mathrm{Hom}(S_t, S_t^\perp))$, dA_t and $*_t$ the area element and the Hodge $*$-operator of (M, g_t), respectively, fixing $g_t \in \mathcal{C}_t$. Thus

$$\frac{d}{dt}_{|t=0}E(S_t, \mathcal{C}_t) = \frac{1}{2}\int_M ((d\dot{S} \wedge *dS) + (dS \wedge \dot{*}dS) + (dS \wedge *d\dot{S})),$$

abbreviating $\frac{d}{dt}_{|t=0}$ by a dot and writing $*$ for $*_0$. Now we verify that

$$(dS \wedge \dot{*}dS) = 0. \tag{4.10}$$

Fix $X \in \Gamma(TM)$ locally never-zero, so that X, JX provides a local frame of TM, for J the canonical complex structure in $(M, \mathcal{C}_\Lambda)$. The 2-form $(dS \wedge \dot{*}dS)$ vanishes if and only if $(dS \wedge \dot{*}dS)(X, JX) = 0$, or, equivalently, $(dS \wedge \dot{*}dS)(X + iJX, X - iJX) = 0$. For each t, let $J_t \in \Gamma(\mathrm{End}(TM))$ be the canonical complex structure in $(M, \mathcal{C}_t)$, writing also J for J_0. Differentiation at $t = 0$ of $*_t dS_t = -(dS_t)J_t$ gives $\dot{*}dS + *d\dot{S} = -(d\dot{S})J - (dS)\dot{J} = *d\dot{S} - (dS)\dot{J}$ and, therefore,

$$\dot{*}dS = -(dS)\dot{J}.$$

Hence equation (4.10) holds if and only if $-(d_{X^{0,1}}S, d_{\dot{J}X^{1,0}}S) + (d_{X^{1,0}}S, d_{\dot{J}X^{0,1}}S) = 0$, for $X^{0,1} := X + iJX$ and $X^{1,0} := X - iJX$. Now differentiation at $t = 0$ of $J_t^2 = -I$ gives

$$\dot{J}J = -J\dot{J}$$

and, consequently, that $\dot{J}$ intertwines the eigenspaces of J: given $X \in \Gamma(TM)$,

$$J(\dot{J}(X \pm iJX)) = -\dot{J}(J(X \pm iJX)) = \pm i\dot{J}(X \pm iJX),$$

respectively, showing that $\dot{J}(T^{1,0}M) \subset T^{0,1}M$, $\dot{J}(T^{0,1}M) \subset T^{1,0}M$. By the conformality of $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$, it follows that $(d_{X^{0,1}}S, d_{\dot{J}X^{1,0}}S) = 0 = (d_{X^{1,0}}S, d_{\dot{J}X^{0,1}}S)$.

We establish (4.10) and, consequently, that

$$\begin{aligned}\frac{d}{dt}_{|t=0} E(S_t, \mathcal{C}_t) &= \frac{1}{2}\int_M ((d\dot{S} \wedge *dS) + (dS \wedge *d\dot{S}))\\ &= \frac{d}{dt}_{|t=0}\int_M \frac{1}{2}(dS_t \wedge *dS_t)\\ &= \frac{d}{dt}_{|t=0} E(S_t, \mathcal{C}_\Lambda),\end{aligned}$$

completing the proof. □

Lemma 4.11 establishes, in particular, that Λ is a Willmore surface as soon as $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$ is harmonic. However, it does not prove Theorem 4.10, as a variation of the central sphere congruence is not necessarily a variation through central sphere congruences.

Lemma 4.12. *Let $(\Lambda_t)_t$ be a variation of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$ and $(S_t)_t$ be the corresponding variation of S through central sphere congruences. For each t, let σ_t be a never-zero section of Λ_t, writing also σ for σ_0. The variational vector fields of $(\sigma_t)_t$ and $(S_t)_t$ are related by*

$$\dot{S}(\sigma) = \pi_{S^\perp}(\dot{\sigma}).$$

Proof. Let $\mathcal{S} : M \times]-\varepsilon, \varepsilon[\to \mathcal{G}$ be defined by $\mathcal{S}(p,t) := S_t(p)$. Under the identification of $S^*T\mathcal{G}$ with $\mathrm{Hom}(S, S^\perp)$ defined by (2.9) for the case $T = S$, given $\xi : M \times]-\varepsilon, \varepsilon[\to \mathbb{R}^{n+1,1}$ such that $\xi_0 := (x \mapsto \xi(x,0))$ is a section of S, we have $d\mathcal{S}_{(p,0)}(u,k)(\xi_0(p)) = \pi_{S^\perp}(d\xi_{(p,0)}(u,k))$, for all $p \in M$, $u \in T_pM$ and $k \in \mathbb{R}$. In particular, for ξ defined by $\xi(p,t) := \sigma_t(p)$, for $u = 0$ and $k = (d/dt)_{t=0}$, we get

$$d(\mathcal{S}^p)_0((d/dt)_{t=0})\sigma(p) = \pi_{S^\perp}(d(\xi^p)_0((d/dt)_{t=0}),$$

for $\mathcal{S}^p := (t \mapsto \mathcal{S}(p,t))$ and $\xi^p := (t \mapsto \xi(p,t))$. Equivalently,

$$(d/dt)_{t=0}\, S_t(p)\sigma(p) = \pi_{S^\perp}((d/dt)_{t=0}\, \sigma_t(p)).$$

□

Remark 4.13. *Lemma 4.12 establishes, in particular, that $\pi_{S^\perp}\dot{\sigma}$ does not depend on the variation $(\sigma_t)_t$ of σ, only on the variation $(\langle\sigma_t\rangle)_t = (\Lambda_t)_t$ of $\langle\sigma\rangle = \Lambda$. Furthermore, given a variation $(\sigma'_t)_t = (\lambda_t\sigma_t)_t$ of σ, with $\lambda_t \in \Gamma(\underline{\mathbb{R}})$ never-zero for all t, the respective variational vector field $\dot{\sigma}'$ relates to $\dot{\sigma}$ by $\dot{\sigma}' = (\frac{d}{dt}_{|t=0}\lambda_t)\sigma'_0 + \lambda_0\dot{\sigma}$ and, therefore,*

$$\dot{\sigma}' = \dot{\sigma} \,\mathrm{mod} \Lambda.$$

Given z a holomorphic chart of $(M, \mathcal{C}_\Lambda)$, we use τ_z to denote the tension field of $S : (M, g_z) \to \mathcal{G}$,

$$\tau_z = \mathrm{tr}_z \nabla^z dS \in \Gamma(S^*T\mathcal{G}),$$

for $\nabla^z dS$ the Hessian of $S : (M, g_z) \to \mathcal{G}$ and tr_z indicating trace computed with respect to $g_z \in \mathcal{C}_\Lambda$. For simplicity, let ∇ denote the pull-back connection on $S^*T\mathcal{G}$ induced by the Levi-Civita connection on $\mathcal{G}$ (when provided with the pseudo-Riemannian structure defined in Section 2.1).

Lemma 4.14. *Given z a holomorphic chart of $(M, \mathcal{C}_\Lambda)$,*

$$4\,\nabla_{\delta_z} S_{\bar{z}} = \tau_z = 4\,\nabla_{\delta_{\bar{z}}} S_z. \tag{4.11}$$

It follows that, under the usual identification $S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp)$,

$$\Lambda^{(1)} \subset \ker \tau_z \tag{4.12}$$

and, consequently,

$$\mathrm{Im}\, \tau_z^t \subset \Lambda. \tag{4.13}$$

Proof. Fix a holomorphic chart $z = x + iy$ of $(M, \mathcal{C}_\Lambda)$. First of all, note that, as δ_x, δ_y is an orthonormal frame of (TM, g_z), we have

$$\tau_z = \nabla_{\delta_x} S_x - dS(\nabla^{g_z}_{\delta_x} \delta_x) + \nabla_{\delta_y} S_y - dS(\nabla^{g_z}_{\delta_y} \delta_y),$$

for ∇^{g_z} the Levi-Civita connection on (M, g_z). On the other hand, for J the canonical complex-structure in $(M, \mathcal{C}_\Lambda)$, (M, g_z, J) is a Kähler manifold and, therefore,

$$J(\nabla^{g_z}_{\delta_x} \delta_x + \nabla^{g_z}_{\delta_y} \delta_y) = \nabla^{g_z}_{\delta_x} J\delta_x + \nabla^{g_z}_{\delta_y} J\delta_y = \nabla^{g_z}_{\delta_x} \delta_y - \nabla^{g_z}_{\delta_y} \delta_x = [\delta_x, \delta_y] = 0.$$

Thus

$$\tau_z = \nabla_{\delta_x} S_x + \nabla_{\delta_y} S_y.$$

By the symmetry of the Hessian,

$$\nabla_{\delta_x} S_y - dS(\nabla^{g_z}_{\delta_x} \delta_y) = \nabla_{\delta_y} S_x - dS(\nabla^{g_z}_{\delta_y} \delta_x),$$

or, equivalently, by the torsion-free property of the Levi-Civita connection,

$$\nabla_{\delta_x} S_y - \nabla_{\delta_y} S_x = dS([\delta_x, \delta_y]) = 0.$$

This establishes (4.11). Next observe that, whilst, for a never-zero section σ of Λ, $(\sigma_{zz}, \sigma) = (\sigma, \sigma_z)_z - (\sigma_z, \sigma_z) = 0$, as well as $(\sigma_{zz}, \sigma_z) = \frac{1}{2}(\sigma_z, \sigma_z)_z = 0$; for the particular case of the normalized section σ^z of Λ with respect to z, we have, furthermore $(\sigma^z_{zz}, \sigma^z_{\bar{z}}) = (\sigma^z_z, \sigma^z_{\bar{z}})_z - (\sigma^z_z, \sigma^z_{z\bar{z}}) = 0$, as $(\sigma^z_z, \sigma^z_{\bar{z}})$ is constant. Hence $\pi_S(\sigma^z_{zz})$ is orthogonal to σ^z, σ^z_z and $\sigma^z_{\bar{z}}$, so that $\pi_S(\sigma^z_{zz}) \in \Gamma(\Lambda)$ and, therefore, $(\pi_S(\sigma^z_{zz}))_{\bar{z}} \in \Gamma(S)$. It

follows that

$$\begin{aligned}\tau_z(\sigma^z_z) &= 4\,(\nabla_{\delta_z}S_{\bar z})\sigma^z_z\\ &= 4\,(\nabla^{S^\perp}_{\delta_z}(S_{\bar z}\sigma^z_z) - S_{\bar z}(\nabla^S_{\delta_z}\sigma^z_z))\\ &= 4\,(\nabla^{S^\perp}_{\delta_z}(\pi_{S^\perp}(\sigma^z_{z\bar z})) - S_{\bar z}(\pi_S(\sigma^z_{zz})))\\ &= -4\,\pi_{S^\perp}((\pi_S(\sigma^z_{zz}))_{\bar z})\\ &= 0.\end{aligned}$$

The orthogonality of $\pi_S(\sigma^z_{zz})$ to σ^z, σ^z_z and $\sigma^z_{\bar z}$ establishes that of $\pi_S(\sigma^z_{\bar z\bar z})$, establishing, ultimately,

$$\tau_z(\sigma^z_{\bar z}) = 4\,(\nabla_{\delta_{\bar z}}S_z)\sigma^z_{\bar z} = -4\,\pi_{S^\perp}((\pi_S(\sigma^z_{\bar z\bar z}))_z) = 0.$$

On the other hand,

$$\tau_z(\sigma^z) = 4\,(\nabla^{S^\perp}_{\delta_z}(\pi_{S^\perp}(\sigma^z_{\bar z})) - \pi_{S^\perp}(\sigma^z_{z\bar z})) = 0.$$

We conclude that $\Lambda^{(1)} \subset \ker\tau_z$ and, consequently, that

$$\operatorname{Im}\tau^t_z \subset (\ker\tau_z)^\perp \cap S \subset (\Lambda^{(1)})^\perp \cap S = \Lambda.$$

$\square$

Now we proceed to the proof of Theorem 4.10.

Proof. Suppose $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$ is harmonic. Then, in particular, given an arbitrary variation $(\Lambda_t)_t$ of Λ through surfaces in the projectivized light-cone, we have

$$\frac{d}{dt}_{|t=0} E(S_t, \mathcal{C}_\Lambda) = 0, \tag{4.14}$$

for the corresponding variation $(S_t)_t$ of S through central sphere congruences. By equation (4.9), we conclude that Λ is a Willmore surface.

Conversely, suppose that Λ is a Willmore surface. Fix a holomorphic chart z of $(M, \mathcal{C}_\Lambda)$. To prove that $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$ is harmonic, we consider the usual identification $S^*T\mathcal{G} \cong \operatorname{Hom}(S, S^\perp)$ and show that $\tau_z \in \Gamma(\operatorname{Hom}(S, S^\perp))$ vanishes. For that, and aiming for a contradiction, suppose that τ_z is non-zero. Then so is $\tau^t_z \in \Gamma(\operatorname{Hom}(S^\perp, \Lambda))$. Fix a never-zero section σ of Λ and a variation $(\sigma_t)_t$ of σ through smooth maps $\sigma_t : M \to \mathcal{L}$ with $\dot\sigma \in \Gamma(\langle\sigma\rangle^\perp) = \Gamma(\Lambda^{(1)} \oplus S^\perp)$ a section of $S^\perp$ such that $\tau^t_z(\pi_{S^\perp}\dot\sigma) = \lambda\sigma$ for some positive $\lambda \in C^\infty(M, \mathbb{R})$. Define a variation of Λ through surfaces in the projectivized light-cone by setting $\Lambda_t := \langle\sigma_t\rangle$, for each t. Let $(S_t)_t$ be the corresponding variation of S through central sphere congruences and $\dot S$ be the corresponding variational vector field. According to Lemma 4.12, $\tau^t_z\,\dot S(\sigma) = \lambda\sigma$. On the other hand, yet again according to (4.13), $\operatorname{tr}(\tau^t_z\,\dot S)$ is simply the component of $\tau^t_z\,\dot S(\sigma)$ with respect to σ. Hence $\operatorname{tr}(\tau^t_z\,\dot S) = \lambda$ is positive. Lastly, the fact that Λ is a Willmore surface intervenes

to establish (4.14). On the other hand, according to equation (4.6),

$$\frac{d}{dt}_{|t=0} E(S_t, \mathcal{C}_\Lambda) = -\int_M (\dot{S}, \tau_z)\,\mathrm{dA}_z = -\int_M \mathrm{tr}(\tau_z^t\,\dot{S})\,\mathrm{dA}_z, \tag{4.15}$$

for dA_z the area element of (M, g_z). It follows that

$$\int_M \mathrm{tr}(\tau_z^t\,\dot{S})\,\mathrm{dA}_z = 0, \tag{4.16}$$

which contradicts the conclusion of the positiveness of $\mathrm{tr}(\tau_z^t\,\dot{S})$, completing the proof.

□

4.5. The Willmore surface equation

Having characterized conformal Willmore surfaces in the projectivized light-cone by the harmonicity of the central sphere congruence, we have, in particular, deduced the Willmore surface equation for a conformal immersion $\Lambda : M \to \mathbb{P}(\mathcal{L})$:

$$d^{\nabla^{S^*T\mathcal{G}}} * dS = 0,$$

for $\nabla^{S^*T\mathcal{G}}$ the pull-back connection on $S^*T\mathcal{G}$ induced by the Levi-Civita connection on $\mathcal{G}$ (when provided with the pseudo-Riemannian structure defined in Section 2.1). It is well-known (see, for example, [**15**]) that the usual identification $S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp)$, of bundles provided with a metric, respects connections,[5] i.e., $\nabla^{S^*T\mathcal{G}}$ consists of the connection induced canonically in $\mathrm{Hom}(S, S^\perp)$ by ∇^S and $\nabla^{S^\perp}$,

$$\nabla^{S^*T\mathcal{G}}\xi = \nabla^{S^\perp} \circ \xi - \xi \circ \nabla^S = \mathcal{D} \circ \xi - \xi \circ \mathcal{D},$$

for all $\xi \in \Gamma(\mathrm{Hom}(S, S^\perp))$. That is,

$$\nabla^{S^*T\mathcal{G}} = \mathcal{D}, \tag{4.17}$$

for the connection induced naturally in $\mathrm{Hom}(S, S^\perp)$ by the connection $\mathcal{D}$ on $\underline{\mathbb{R}}^{n+1,1}$. Note that the connection induced naturally in $S \wedge S^\perp$ by $\mathcal{D}$ coincides with the one induced naturally by ∇^S and $\nabla^{S^\perp}$. By (4.4), we conclude that, under the usual identification

$$S^*T\mathcal{G} \cong \mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp, \tag{4.18}$$

of bundles provided with a metric and a connection, $d^{\mathcal{D}} * \mathcal{N} = 0$, or, equivalently (cf. (2.25)), $d{*}\mathcal{N} = 0$, provides a characterization of Willmore surfaces in the projectivized light-cone. In view of $\mathcal{N}^{1,0} = \frac{1}{2}(\mathcal{N} + i * \mathcal{N})$ (and, therefore, $\mathcal{N}^{0,1} = \frac{1}{2}(\mathcal{N} - i * \mathcal{N})$), Codazzi equation establishes

$$d^{\mathcal{D}}\mathcal{N}^{1,0} = \frac{i}{2}\, d^{\mathcal{D}} * \mathcal{N} = -d^{\mathcal{D}}\mathcal{N}^{0,1}. \tag{4.19}$$

[5]This is the particular case $\phi = S$ and $\mathcal{G} = \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ of a fact regarding a general map $\phi : M \to \mathcal{G}$ into a general Grassmannian $\mathcal{G} = Gr_{(r,s)}(\mathbb{R}^{p,q})$.

It follows that:

Theorem 4.15. *Willmore surfaces in the projectivized light-cone are characterized, equivalently, by any of the following equations:*

i) $d * \mathcal{N} = 0$;
ii) $d^{\mathcal{D}} * \mathcal{N} = 0$;
iii) $d^{\mathcal{D}} \mathcal{N}^{1,0} = 0$;
iv) $d^{\mathcal{D}} \mathcal{N}^{0,1} = 0$.

Remark 4.16. *According to Lemma 4.14, together with* (4.17)*, the harmonicity of the central sphere congruence* $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$ *of* Λ *can be characterized by* $\mathcal{D}_{\delta_z} S_{\bar{z}} = 0$*, or, equivalently,* $(\mathcal{D}_{\delta_z} S_{\bar{z}})\, \sigma_{z\bar{z}} = 0$*, fixing a never-zero section* σ *of* Λ *and a holomorphic chart* z *of* $(M, \mathcal{C}_\Lambda)$*.*

Remark 4.17. *Let* $v_\infty \in \mathbb{R}^{n+1,1}$ *be non-zero and* $\sigma_\infty : M \to S_{v_\infty}$ *be the surface defined by* Λ *in the space-form* S_{v_∞}*. Let* Δ_∞ *be the Laplacian in* N_∞ *and* $\tilde{A}_\infty := A^*_\infty \circ A_\infty$*, for* A_∞ *mapping a unit* $\xi \in \Gamma(N_\infty)$ *to* A^ξ_∞*, the shape operator of* σ_∞ *with respect to* ξ*. Cf.* [**57**]*,*

$$\Delta_\infty \mathcal{H}_\infty - 2|\mathcal{H}_\infty|^2 \mathcal{H}_\infty + \tilde{A}_\infty(\mathcal{H}_\infty) = 0$$

is a Willmore surface equation for σ_∞*, providing, therefore, yet another Willmore surface equation for* Λ*. One which takes us out of the path of this text, though.*

We dedicate what is left in this section to contemplating the variational Willmore energy, supposing M is compact. Let $(\Lambda_t)_t$ be a variation of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$ and $\dot{\mathcal{W}}$ be the corresponding variational Willmore energy,

$$\dot{\mathcal{W}} = \frac{d}{dt}_{|t=0} \mathcal{W}(\Lambda_t).$$

For each t, let σ_t be a never-zero section of Λ_t, writing also σ for σ_0. Let $\dot{\sigma}$ be the variational vector field of the variation $(\sigma_t)_t$. Differentiation of $(\sigma_t, \sigma_t) = 0$ establishes $(\sigma, \dot{\sigma}) = 0$. In view of Remark 4.13, define

$$\dot{\Lambda} \in \Gamma(\mathrm{Hom}(\Lambda, \Lambda^\perp/\Lambda)) \tag{4.20}$$

by $\dot{\Lambda}\, \sigma := \dot{\sigma} \,\mathrm{mod}\Lambda$. The notation is not casual. In fact, under the isomorphism

$$d\pi_\sigma : \Lambda^\perp/\Lambda \cong T_\Lambda \mathbb{P}(\mathcal{L})$$

(cf. (1.3)), the variational vector field of $(\Lambda_t)_t$ is $\dot{\sigma} \,\mathrm{mod}\Lambda \in \Gamma(\mathrm{Hom}(\Lambda^\perp/\Lambda))$. Set

$$\nu := \pi \dot{\Lambda} \in \Gamma(\mathrm{Hom}(\Lambda, S^\perp)),$$

for the canonical projection

$$\pi : \Gamma(\mathrm{Hom}(\Lambda, \Lambda^\perp/\Lambda) = \mathrm{Hom}(\Lambda, \Lambda^{(1)}/\Lambda)) \oplus \Gamma(\mathrm{Hom}(\Lambda, S^\perp)) \to \Gamma(\mathrm{Hom}(\Lambda, S^\perp)).$$

Observe that

$$\nu\sigma = \pi_{S^\perp}(\dot{\Lambda}\sigma) = \pi_{S^\perp}\dot{\sigma}.$$

Having said so, let $(S_t)_t$ be the variation of S through central sphere congruences corresponding to the variation $(\Lambda_t)_t$ of Λ and $\dot{S}$ be the corresponding variational central sphere congruence. Fix a holomorphic chart z of $(M, \mathcal{C}_\Lambda)$. According to Lemma 4.11, together with (4.15),

$$\dot{\mathcal{W}} = -\int_M \mathrm{tr}(\tau_z^t\, \dot{S})\, \mathrm{dA}_z$$

and, therefore, by Lemma 4.14, followed by Lemma 4.12,

$$\dot{\mathcal{W}} = -\int_M (\tau_z^t\, \dot{S}\,\sigma, \sigma_{z\bar{z}})(\sigma, \sigma_{z\bar{z}})^{-1}\, \mathrm{dA}_z = -\int_M (\dot{\sigma}, \tau_z\sigma_{z\bar{z}})(\sigma, \sigma_{z\bar{z}})^{-1} dA_z.$$

The skew-symmetry of $\tau_z \in \Gamma(\mathrm{Hom}(S, S^\perp) \cong S \wedge S^\perp)$ establishes then

$$\dot{\mathcal{W}} = \int_M (\tau_z \pi_{S^\perp}\dot{\sigma}, \sigma_{z\bar{z}})(\sigma, \sigma_{z\bar{z}})^{-1} dA_z.$$

Let $*_z$ be the Hodge $*$-operator on forms over (M, g_z). According to equation (4.7), $d^{\nabla^{S^*T\mathcal{G}}} * dS = -*_z\, \tau_z$ and, therefore, under the identification (4.18),

$$\tau_z = *_z\, d^{\mathcal{D}} * \mathcal{N} \in \Gamma(S \wedge S^\perp). \tag{4.21}$$

It follows that

$$\dot{\mathcal{W}} = \int_M ((*_z\, d^{\mathcal{D}} * \mathcal{N})\nu\sigma, \sigma_{z\bar{z}})(\sigma, \sigma_{z\bar{z}})^{-1} dA_z.$$

As we know, since $\mathcal{N}$ takes values in $S \wedge S^\perp$ and S and $S^\perp$ are $\mathcal{D}$-parallel, the 2-form $d^{\mathcal{D}} * \mathcal{N}$ takes values in $S \wedge S^\perp$, and so does, therefore, $*_z\, d^{\mathcal{D}} * \mathcal{N}$. In view of equations (4.12) and (4.21), we conclude, furthermore, that

$$*_z\ d^{\mathcal{D}} * \mathcal{N} \in \Omega^0(\Lambda \wedge S^\perp). \tag{4.22}$$

Hence $(*_z\, d^{\mathcal{D}} * \mathcal{N}) \circ \nu \in \Gamma(\mathrm{End}(\Lambda))$ and

$$\begin{aligned}
\dot{\mathcal{W}} &= \int_M \mathrm{tr}\,((*_z\, d^{\mathcal{D}} * \mathcal{N}) \circ \nu)\, dA_z \\
&= \int_M (\nu, (*_z\, d^{\mathcal{D}} * \mathcal{N})^t)\, dA_z \\
&= -\int_M (*_z\, d^{\mathcal{D}} * \mathcal{N}, \nu)\, dA_z
\end{aligned}$$

and, ultimately,

$$\dot{\mathcal{W}} = -\int_M ((d^{\mathcal{D}} * \mathcal{N}) \wedge \nu).$$

As $*_z\, d^{\mathcal{D}} * \mathcal{N}$ vanishes on $\Lambda^{(1)}$, we have

$$(*_z\, d^{\mathcal{D}} * \mathcal{N}, \dot{\Lambda} - \nu) = \mathrm{tr}\,((*_z\, d^{\mathcal{D}} * \mathcal{N})^T(\dot{\Lambda} - \nu)) = -\mathrm{tr}\,(*_z\, d^{\mathcal{D}} * \mathcal{N} \circ (\dot{\Lambda} - \nu)) = 0$$

and we conclude that the variational Willmore energy relates to the variational surface by

$$\dot{\mathcal{W}} = -\int_M ((d^{\mathcal{D}} * \mathcal{N}) \wedge \dot{\Lambda}). \tag{4.23}$$

As a final remark, note that, according to Lemma 4.11 and (4.15), on the other hand, $\dot{\mathcal{W}} = -\int_M (\dot{S}, \tau_z)\, \mathrm{dA}_z$ and, therefore, by (4.21),

$$\dot{\mathcal{W}} = -\int_M ((d^{\mathcal{D}} * \mathcal{N}) \wedge \dot{S}).$$

4.6. Willmore surfaces under change of flat metric connection

Let Λ be a null line subbundle of the trivial bundle $M \times \mathbb{R}^{n+1,1}$, not necessarily defining an immersion into $\mathbb{P}(\mathcal{L})$. Let $\tilde{d}$ be a flat metric connection on $\underline{\mathbb{R}}^{n+1,1}$.

Definition 4.18. *Suppose Λ is a $\tilde{d}$-surface. Λ is said to be a* Willmore $\tilde{d}$-surface *if*

$$d^{\mathcal{D}^{\tilde{d}}} *_{\tilde{d}} \mathcal{N}^{\tilde{d}} = 0,$$

where $*_{\tilde{d}}$ denotes the Hodge $*$-operator on 1-forms over $(M, \mathcal{C}^{\tilde{d}}_{\Lambda})$.

This definition is a generalization of the characterization of a Willmore surface in the projectivized light-cone provided by $d^{\mathcal{D}} * \mathcal{N} = 0$, corresponding to the particular case $\tilde{d} = d$ and M is compact.

Let $\tilde{\phi} : (\underline{\mathbb{R}}^{n+1,1}, \tilde{d}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ be an isomorphism. As observed in Section 3, Λ is a $\tilde{d}$-surface if and only if $\tilde{\phi}\Lambda$ is a surface. Furthermore:

Proposition 4.19. *Suppose Λ is a $\tilde{d}$-surface (or, equivalently, $\tilde{\phi}\Lambda$ is a surface). In that case, Λ is a Willmore $\tilde{d}$-surface if and only if $\tilde{\phi}\Lambda$ is a Willmore surface.*

Proof. Set $\tilde{\Lambda} = \tilde{\phi}\Lambda$. By (3.4), relating $\mathcal{D}_{\tilde{\Lambda}}$ to $\mathcal{D}^{\tilde{d}}$ and $\mathcal{N}_{\tilde{\Lambda}}$ to $\mathcal{N}^{\tilde{d}}$, we have, given $X, Y \in \Gamma(TM)$,

$$d^{\mathcal{D}_{\tilde{\Lambda}}} *_{\tilde{d}} \mathcal{N}_{\tilde{\Lambda}}(X,Y) = d^{\mathcal{D}_{\tilde{\Lambda}}}\, \tilde{\phi}\, (*_{\tilde{d}} \mathcal{N}^{\tilde{d}})\, \tilde{\phi}^{-1}(X,Y) = \tilde{\phi}\; (d^{\mathcal{D}^{\tilde{d}}} *_{\tilde{d}} \mathcal{N}^{\tilde{d}}(X,Y))\, \tilde{\phi}^{-1}.$$

The fact that $\mathcal{C}^{\tilde{d}}_{\Lambda} = \mathcal{C}_{\tilde{\Lambda}}$ (cf. (3.2)) completes the proof. □

4.7. Spectral deformation of Willmore surfaces

Let $\phi : M \to Gr_{(r,s)}(\mathbb{R}^{p,q})$ be a map into the Grassmannian $Gr_{(r,s)}(\mathbb{R}^{p,q})$. Let π_ϕ and $\pi_{\phi^\perp}$ be the orthogonal projections of $\underline{\mathbb{R}}^{p,q}$ onto ϕ and $\phi^\perp$, respectively. Provide ϕ and $\phi^\perp$ with the connections $\nabla^\phi := \pi_\phi \circ d \circ \pi_\phi$ and $\nabla^{\phi^\perp} := \pi_{\phi^\perp} \circ d \circ \pi_{\phi^\perp}$, respectively. Set $\mathcal{D}_\phi := \nabla^\phi + \nabla^{\phi^\perp}$ and $\mathcal{N}_\phi := d - \mathcal{D}_\phi$. Under the standard identification $\phi^* TGr_{(r,s)}(\mathbb{R}^{p,q}) \cong \mathrm{Hom}(\phi, \phi^\perp) \cong \phi \wedge \phi^\perp$, of bundles provided with a metric and a connection, $d\phi = \mathcal{N}_\phi$, so that the harmonicity of ϕ with respect to a given conformal structure in M, in the case M is compact, is characterized by $d^{\mathcal{D}_\phi} * \mathcal{N}_\phi = 0$ (noting that the

connection induced naturally in $\phi \wedge \phi^\perp$ by ∇^ϕ and $\nabla^{\phi^\perp}$ coincides with the one induced naturally by $\mathcal{D}_\phi$). K. Uhlenbeck [**56**] proved that the harmonicity of ϕ is characterized, equivalently, by the flatness of the metric connection $d^\lambda_\phi := \mathcal{D}_\phi + \lambda^{-1}\mathcal{N}^{1,0}_\phi + \lambda\mathcal{N}^{0,1}_\phi$, for each $\lambda \in S^1$. Furthermore, such loop of flat metric connections gives rise to a S^1-family of harmonic maps into $Gr_{(r,s)}(\mathbb{R}^{p,q})$, cf. [**56**]. Harmonic maps into Grassmannian manifolds come in S^1-families. In this section, we show that, if $S : M \to \mathcal{G}$ is harmonic, then the S^1-deformation of S defined by the loop of flat metric connections d^λ_S, with $\lambda \in S^1$, is the family of (harmonic) central sphere congruences corresponding to the S^1-deformation of Λ defined by the loop d^λ_S. The characterization of Willmore surfaces in space-forms in terms of the harmonicity of the central sphere congruence gives rise, in this way, to a spectral deformation of Willmore surfaces in space-forms. As we shall verify in section 6.4.1 below, this deformation coincides, up to reparametrization, with the one presented in [**14**].

Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone. Consider M provided with the conformal structure $\mathcal{C}_\Lambda$. For each $\lambda \in S^1$, define a connection on $\underline{\mathbb{R}}^{n+1,1}$ by

$$d^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1},$$

noting that, d^λ is, indeed, real: $\mathcal{D}$ and $\mathcal{N}$ are real and, as λ is unit, $\overline{\lambda} = \lambda^{-1}$, so that, given $\mu \in \Gamma(\underline{\mathbb{R}}^{n+1,1})$, $\overline{d^\lambda \mu} = d^\lambda \mu$.

The next result is, in view of Theorem 4.10, the particular case $\phi = S$ of the characterization of the harmonicity of a map $\phi : M \to \mathcal{G}$ in terms of the flatness of the S^1-family of metric connections d^λ_ϕ, by K. Uhlenbeck [**56**].

Theorem 4.20. *Λ is a Willmore surface if and only if d^λ is a flat connection, for each $\lambda \in S^1$.*

Proof. According to (2.24), and having in consideration that there are no non-zero $(2,0)$- or $(0,2)$-forms over a surface, the curvature tensor of d^λ is given by

$$R^{d^\lambda} = R^{\mathcal{D}} + \lambda^{-1} d^{\mathcal{D}}\mathcal{N}^{1,0} + \lambda\, d^{\mathcal{D}}\mathcal{N}^{0,1} + \frac{1}{2}\,[\mathcal{N} \wedge \mathcal{N}].$$

By Gauss-Ricci and Codazzi equations, it follows that $R^{d^\lambda} = (\lambda^{-1} - \lambda)\, d^{\mathcal{D}}\mathcal{N}^{1,0}$. We conclude that d^λ is flat for all λ in S^1 if and only if $d^{\mathcal{D}}\mathcal{N}^{1,0} = 0$, which, according to Theorem 4.15, completes the proof. $\square$

Since $\mathcal{N}$ is skew-symmetric, the fact that $\mathcal{D}$ is a metric connection ensures that so is d^λ. Therefore, if Λ is a Willmore surface, the family of connections d^λ, with $\lambda \in S^1$, consists of a S^1-family of flat metric connections, defining then a S^1-family of transformations of Λ, by setting, for each $\lambda \in S^1$,

$$\Lambda_\lambda := \phi_{d^\lambda}\Lambda,$$

for some isomorphism $\phi_{d^\lambda}:(\underline{\mathbb{R}}^{n+1,1},d^\lambda)\to(\underline{\mathbb{R}}^{n+1,1},d)$. Observe that, for each $\lambda\in S^1$, Λ_λ consists of a transformation of Λ into another surface. In fact, given a never-zero section σ of Λ, we have

$$d^\lambda\sigma=\mathcal{D}\sigma=d\sigma, \tag{4.24}$$

and, therefore, $\Lambda^{(1)}_{d^\lambda}=\Lambda^{(1)}$, showing that, Λ is a d^λ-surface for all $\lambda\in S^1$. Furthermore:

Theorem 4.21. *If Λ is a Willmore surface, then so is the transformation Λ_λ of Λ defined by the flat metric connection d^λ, for each $\lambda\in S^1$.*

Proof. Suppose Λ is a Willmore surface, in which case such a transformation Λ_λ of Λ is defined. Fix $\lambda\in S^1$ and σ a never-zero section of Λ. First of all, note that, according to (4.24), $g_\sigma^{d^\lambda}=g_\sigma$ and, therefore, $\mathcal{C}_\Lambda^{d^\lambda}=\mathcal{C}_\Lambda$. For simplicity, write S^λ, $\mathcal{D}^\lambda$ and $\mathcal{N}^\lambda$ for, respectively, S^{d^λ}, $\mathcal{D}^{d^\lambda}$ and $\mathcal{N}^{d^\lambda}$. The proof will consist of showing that $d^{\mathcal{D}^\lambda}*\mathcal{N}^\lambda=0$, for $*$ the Hodge $*$-operator on 1-forms over $(M,\mathcal{C}_\Lambda)$. The result will then follow from Proposition 4.19.

The crucial observation is that the d^λ-central sphere congruence of Λ coincides with the central sphere congruence of Λ,

$$S^\lambda=S, \tag{4.25}$$

as we shall see next. For that, it is enough to fix an orthonormal frame $(e_i)_i$ of TM with respect to g_σ and to show that $\sum_i d^\lambda_{e_i}d_{e_i}\sigma=\sum_i d_{e_i}d_{e_i}\sigma$, or, equivalently, that $\sum_i(\lambda^{-1}\mathcal{N}^{1,0}_{e_i}d_{e_i}\sigma+\lambda\mathcal{N}^{0,1}_{e_i}d_{e_i}\sigma)=0$, having in consideration that $\sum_i\mathcal{N}_{e_i}d_{e_i}\sigma=0$. Set $Z:=e_1-i\,e_2$. Note that, given a 2-tensor T on M, $\sum_i T(e_i,e_i)=\frac{1}{2}\,(T(Z,\overline{Z})+T(\overline{Z},Z))$, so that, in particular, if T is symmetric, $T(Z,\overline{Z})=\sum_i T(e_i,e_i)=T(\overline{Z},Z)$. Note that the 2-tensor $((X,Y)\mapsto\mathcal{N}_X(d_Y\sigma))$ is symmetric: given $X,Y\in\Gamma(TM)$,

$$\mathcal{N}_Y(d_X\sigma)-\mathcal{N}_X(d_Y\sigma)=-\pi_{S^\perp}(d_{[X,Y]}\sigma)=0.$$

Choosing the frame (e_1,e_2) to be direct, we have $Je_1=e_2$ and $Je_2=-e_1$, for J the canonical complex structure in $(M,\mathcal{C}_\Lambda)$, and, therefore, $Z\in\Gamma(T^{1,0})$. It follows, on the one hand, that

$$\sum_i\mathcal{N}^{1,0}_{e_i}d_{e_i}\sigma=\frac{1}{2}\,(\mathcal{N}^{1,0}_Z d_{\overline{Z}}\sigma+\mathcal{N}^{1,0}_{\overline{Z}}d_Z\,\sigma)=\frac{1}{2}\,\mathcal{N}^{1,0}_Z d_{\overline{Z}}\sigma,$$

and, on the other hand,

$$\mathcal{N}^{1,0}_Z d_{\overline{Z}}\sigma=\mathcal{N}_{e_1}d_{e_1}\sigma+i\mathcal{N}_{e_1}d_{e_2}\sigma-i\mathcal{N}_{e_2}d_{e_1}\sigma+\mathcal{N}_{e_2}d_{e_2}\sigma=\sum_i\mathcal{N}^{1,0}_{e_i}d_{e_i}\sigma.$$

Analogously,

$$\frac{1}{2}\,\mathcal{N}^{0,1}_{\overline{Z}}d_Z\,\sigma=\sum_i\mathcal{N}^{0,1}_{e_i}d_{e_i}\sigma=\mathcal{N}^{0,1}_{\overline{Z}}d_Z\,\sigma.$$

Thus

$$\sum_i \mathcal{N}^{1,0}_{e_i} d_{e_i}\sigma = 0 = \sum_i \mathcal{N}^{0,1}_{e_i} d_{e_i}\sigma,$$

completing the verification of (4.25). Now note that, as $\mathcal{N}$ intertwines S and $S^\perp$, we have

$$\pi_S \circ d^\lambda \circ \pi_S = \pi_S \circ d \circ \pi_S, \quad \pi_{S^\perp} \circ d^\lambda \circ \pi_{S^\perp} = \pi_{S^\perp} \circ d \circ \pi_{S^\perp}.$$

We conclude that

$$\mathcal{D}^\lambda = \mathcal{D}$$

and, consequently,

$$\mathcal{N}^\lambda = \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1}.$$

Hence

$$\begin{aligned} d^{\mathcal{D}^\lambda} * \mathcal{N}^\lambda &= \lambda^{-1} d^{\mathcal{D}} * \mathcal{N}^{1,0} + \lambda\, d^{\mathcal{D}} * \mathcal{N}^{0,1} = \\ &= -i\lambda^{-1} d^{\mathcal{D}}\mathcal{N}^{1,0} + i\lambda\, d^{\mathcal{D}}\mathcal{N}^{0,1} = \\ &= -i(\lambda^{-1} + \lambda) d^{\mathcal{D}}\mathcal{N}^{1,0}, \end{aligned}$$

which, according to Theorem 4.15, completes the proof. □

The harmonicity of $S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$, characterized, cf. K. Uhlenbeck [**56**], by the flatness of d^λ_S (spelt out with respect to $\mathcal{C}_\Lambda$), for all $\lambda \in S^1$, establishes, equivalently, the flatness of

$$d^\mu_{\phi_{d^\lambda} S} = \phi_{d^\lambda} \circ d^{\lambda\mu}_S \circ \phi^{-1}_{d^\lambda},$$

for all $\lambda, \mu \in S^1$, or, equivalently, the harmonicity of $\phi_{d^\lambda} S : (M, \mathcal{C}_\Lambda) \to \mathcal{G}$, for all λ. According to (4.25), the S^1-deformation $\phi_{d^\lambda} S$ of S is the family of central sphere congruences corresponding to the S^1-deformation $\phi_{d^\lambda}\Lambda$ of Λ. On the other hand, in view of (4.24), $g_{\phi_{d^\lambda}\Lambda} = g_\sigma$ and, therefore,

$$\mathcal{C}_{\phi_{d^\lambda}\Lambda} = \mathcal{C}_\Lambda.$$

Hence the harmonicity of S with respect to $\mathcal{C}_\Lambda$ establishes the harmonicity of $\phi_{d^\lambda} S$ with respect to $\mathcal{C}_{\phi_{d^\lambda}\Lambda}$, for all λ. The loop of flat metric connections d^λ defines, in this way, a conformal S^1-deformation of a Willmore surface into a family of Willmore surfaces. As we shall verify in Section 6.4.1 below, this deformation coincides, up to reparametrization, with the one presented in [**14**].

CHAPTER 5

The constrained Willmore surface equation

In this chapter, we introduce constrained Willmore surfaces, the generalization of Willmore surfaces that arises when we consider extremals of the Willmore functional with respect to infinitesimally conformal variations,[1] rather than with respect to all variations. The topic is mentioned very briefly in [**59**]. Some results on constrained Willmore surfaces are contained in [**51**], [**14**], [**7**] and [**11**]. Constrained Willmore surfaces in space forms form a Möbius invariant class of surfaces, with strong links to the theory of integrable systems, which we shall explore throughout this work. F. Burstall et al. [**14**] established a manifestly conformally invariant characterization of constrained Willmore surfaces in space-forms, which, in particular, extended the concept of constrained Willmore to surfaces that are not necessarily compact. This chapter is dedicated to deriving the characterization mentioned above, or rather a reformulation of it by F. Burstall and D. Calderbank [**11**], from the variational problem.

Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone. Provide M with the conformal structure $\mathcal{C}_\Lambda$, induced by Λ. Suppose M is compact.

5.1. Constrained Willmore surfaces: definition and examples

Constrained Willmore surfaces are the critical points of the Willmore functional with respect to infinitesimally conformal variations. They form a Möbius invariant class of surfaces. Willmore surfaces and constant mean curvature surfaces in 3-dimensional space-forms, and so their Möbius transforms, are examples of constrained Willmore surfaces.

A variation $(\Lambda_t)_t$ of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$ is said to be a *conformal variation* if it preserves the conformal structure induced in M, or, equivalently, it preserves the isotropy of $T^{1,0}M$ (respectively, $T^{0,1}M$), i.e., fixing $Z \in \Gamma(T^{1,0}M)$ (respectively, $Z \in \Gamma(T^{0,1}M)$), locally never-zero, and, for each t, g_t in the conformal class of metrics induced in M by Λ_t,

$$g_t(Z, Z) = 0.$$

[1] To which references as *conformal variations* can be found in the literature.

The constraint on the conformal structure we are interested in is weaker than conformality.

Definition 5.1. *A variation $(\Lambda_t)_t$ of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$ is said to be an* infinitesimally conformal variation *if, fixing $Z \in \Gamma(T^{1,0}M)$ (respectively, $Z \in \Gamma(T^{0,1}M)$), locally never-zero, and, for each t, g_t in the conformal class of metrics induced in M by Λ_t, we have*

$$\frac{d}{dt}_{|t=0} g_t(Z,Z) = 0.$$

Note that this is, indeed, a good definition, as, given $\lambda = (t \mapsto \lambda_t)$, with $\lambda_t : M \to \mathbb{R}$ positive for each t,

$$\frac{d}{dt}_{|t=0} \lambda_t g_t(Z,Z) = \lambda'(0) g(Z,Z) + \lambda(0) \frac{d}{dt}_{|t=0} g_t(Z,Z),$$

with $g \in \mathcal{C}_\Lambda$, so that $\frac{d}{dt}_{|t=0} \lambda_t g_t(Z,Z) = \lambda(0) \frac{d}{dt}_{|t=0} g_t(Z,Z)$, which vanishes if and only if $\frac{d}{dt}_{|t=0} g_t(Z,Z)$ does.

Definition 5.2. *The surface Λ is said to be a* constrained Willmore surface *if*

$$\frac{d}{dt}_{|t=0} W(\Lambda_t) = 0$$

for every infinitesimally conformal variation $(\Lambda_t)_t$ of Λ through immersions of M in $\mathbb{P}(\mathcal{L})$.

It is immediate from Theorem 4.1 that conformal diffeomorphisms transform constrained Willmore surfaces into constrained Willmore surfaces.

Theorem 5.3. *The class of constrained Willmore surfaces is Möbius invariant.*

In particular:

Proposition 5.4. *Λ is a constrained Willmore surface if and only if, fixing a non-zero $v_\infty \in \mathbb{R}^{n+1,1}$, so is the surface in S_{v_∞} defined by Λ.*

In Section 6.3.1, we extend the concept of constrained Willmore surface to surfaces that are, in particular, not necessarily compact.

Willmore surfaces are, obviously, examples of constrained Willmore surfaces. But there are more: constant mean curvature (CMC) surfaces in 3-dimensional space-forms are constrained Willmore (and also isothermic). Section 8.2 is dedicated to this special class of surfaces. We believe one can obtain non-isothermic, non-Willmore constrained Willmore surfaces as *Bäcklund transforms* of non-minimal CMC surfaces in space-forms, following Section 8.1.5 below, but this is not clear, though (it shall be the subject of further work). Section 7.2.2 is dedicated to another class of constrained Willmore surfaces, that of codimension 2 surfaces with holomorphic mean curvature

vector in space-forms.

5.2. The Hopf differential and the Schwarzian derivative

In [**14**], a characterization of constrained Willmore surfaces, isothermic surfaces and constant mean curvature surfaces in space-forms in terms of the Hopf differential and the Schwarzian derivative is established. It is a uniform characterization of these three classes of surfaces and, for this reason, it will be presented in this text, in parallel to what is our main approach. In this section, we introduce the Hopf differential and the Schwarzian derivative, cf. [**14**].

Fix $z = x + iy$ a holomorphic chart of M and consider σ^z, the normalized section of Λ with respect to z. Write

$$\sigma^z_{zz} = a\sigma^z + b\sigma^z_z + c\sigma^z_{\bar{z}} + d\sigma^z_{z\bar{z}} + \pi_{S^\perp}\sigma^z_{zz},$$

with $a, b, c, d \in \Gamma(\underline{\mathbb{C}})$. By the orthogonality relations of the frame $\{\sigma, \sigma_z, \sigma_{\bar{z}}, \sigma_{z\bar{z}}\}$, we have

$$b\frac{1}{2} = (\sigma^z_{zz}, \sigma^z_{\bar{z}}) = (\sigma^z_z, \sigma^z_{\bar{z}})_z - (\sigma^z_z, \sigma_{z\bar{z}}) = 0$$

and, therefore, $b = 0$; on the other hand,

$$c\frac{1}{2} = (\sigma^z_{zz}, \sigma^z_z) = \frac{1}{2}\,(\sigma^z_z, \sigma^z_z)_z = 0$$

and, therefore, $c = 0$; and

$$-d\frac{1}{2} = (\sigma^z_{zz}, \sigma^z) = (\sigma^z_z, \sigma^z)_z - (\sigma^z_z, \sigma^z_z) = 0,$$

showing that $d = 0$. Thus σ^z_{zz} satisfies an equation

$$\sigma^z_{zz} = -\frac{1}{2}\,c^z\sigma^z + k^z, \tag{5.1}$$

defining a complex function

$$c^z := 4(\sigma^z_{zz}, \sigma^z_{z\bar{z}}) \in \Gamma(\underline{\mathbb{C}}),$$

the *Schwarzian derivative* with respect to z, and a section

$$k^z := \pi_{S^\perp}(\sigma^z_{zz}) \in \Gamma(S^\perp),$$

of the complexification of the normal bundle to the central sphere congruence of Λ, called the *Hopf differential* of Λ with respect to z.[2]

It is useful to understand how the Hopf differential changes under a change of holomorphic coordinate. Let ω be another holomorphic chart of M. Following equation

[2]The terminology for the latter is motivated by the relation established in Lemma A.2 below, in Appendix A.

(1.10), we have

$$\sigma^w = \mid \omega_z \mid \sigma^z,$$

so that $\sigma^\omega_\omega = \omega_z^{-1} \mid \omega_z \mid_z \sigma^z + \omega_z^{-1} \mid \omega_z \mid \sigma^z_z$ and, consequently,

$$\sigma^\omega_{\omega\omega} = \frac{\mid \omega_z \mid}{\omega_z^2}\left(\left(\frac{\omega_{zz}}{2\omega_z}\right)_z - \left(\frac{\omega_{zz}}{2\omega_z}\right)^2 - \frac{1}{2}c^z\right)\sigma^z + \frac{\mid \omega_z \mid}{\omega_z^2}k^z.$$

Hence

$$k^\omega = \frac{\mid \omega_z \mid}{\omega_z^2} k^z \tag{5.2}$$

We complete this section with a fundamental result of conformal surface theory. As established in [**11**],

Lemma 5.5. *If $\Lambda_1, \Lambda_2 : M \to \mathbb{P}(\mathcal{L}) \cong S^n$ are two conformal immersions inducing the same Hopf differential, Schwarzian derivative and normal connection $\nabla^{S^\perp}$, then there is a Möbius transformation T such that $\Lambda_2 = T\Lambda_1$. (In the particular case of codimension 1 (i.e., $n = 3$), the condition on the normal connection is vacuous.)*

5.3. The Euler-Lagrange constrained Willmore surface equation

F. Burstall et al. [**14**] established a manifestly conformally invariant characterization of constrained Willmore surfaces in space-forms, which, in particular, extended the concept of constrained Willmore to surfaces that are not necessarily compact. In this section, we derive, from the variational problem, a reformulation of this characterization, due to F. Burstall and D. Calderbank [**11**]. The argument consists of a generalization to n-space of the argument presented in [**7**] for the particular case of $n = 3$.

As established in [**11**]:

Theorem 5.6. *The surface Λ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with*

$$d^{\mathcal{D}} q = 0 \tag{5.3}$$

such that

$$d^{\mathcal{D}} * \mathcal{N} = 2\,[q \wedge *\mathcal{N}]. \tag{5.4}$$

*In this case, we may refer to Λ as, specifically, a q-*constrained Willmore surface *and to q as a* [Lagrange] multiplier[3] *to Λ.*

[3]Named after the method of Lagrange multipliers for finding the critical points of a function subject to a constraint.

Willmore surfaces are the 0-constrained Willmore surfaces. The zero multiplier is not necessarily the only multiplier to a constrained Willmore surface with no constraint on the conformal structure. The uniqueness of multiplier is discussed in section 8.1.4.

Remark 5.7. *Let q be a 1-form with values in $\Lambda\wedge\Lambda^{(1)}$. According to (4.22),*

$$d^{\mathcal{D}} * \mathcal{N} \in \Omega^2(\Lambda\wedge S^{\perp}). \tag{5.5}$$

On the other hand, in view of (2.17), we conclude from (2.28) that

$$[q\wedge *\mathcal{N}] \in \Omega^2(\Lambda\wedge S^{\perp}). \tag{5.6}$$

In particular, both $d^{\mathcal{D}} * \mathcal{N}$ and $[q\wedge *\mathcal{N}]$ vanish on $\Lambda^{(1)}$ and are, therefore, determined by the respective restrictions to $\langle u\rangle\oplus S^{\perp}$, fixing $u\in S\backslash\Lambda^{\perp}$ (in particular, for $u=\sigma_{z\bar{z}}$, fixing $\sigma\in\Gamma(\Lambda)$ never-zero and z a holomorphic chart of M). Equation (2.15) establishes, furthermore, that both $d^{\mathcal{D}} * \mathcal{N}$ and $[q\wedge *\mathcal{N}]$ are determined by the respective restrictions to $\langle u\rangle$, fixing $u\in S\backslash\Lambda^{\perp}$. In particular, equation (5.4) holds if and only if, fixing such a u, $(d^{\mathcal{D}} * \mathcal{N})u = 2\,[q\wedge *\mathcal{N}]u$.

The proof of Theorem 5.6 we present follows essentially from the so called Weyl's lemma (see, for example, [**44**], §9), which, in particular, establishes the image of the operator $\bar{\partial}$ as the orthogonal space to the vector space of the holomorphic quadratic differentials, with respect to some non-degenerate pairing. We start by establishing a $1-1$ correspondence between quadratic differentials and real 1-forms taking values in $\Lambda\wedge\Lambda^{(1)}$ whose $(1,0)$-part takes values in $\Lambda\wedge\Lambda^{0,1}$ (for $\Lambda^{0,1}$ as defined next). We verify that condition (5.3) on q forces $q^{1,0}$ to take values in $\Lambda\wedge\Lambda^{0,1}$ and that, under the correspondence above, condition (5.3) is equivalent to the holomorphicity of the quadratic differential q. And then we proceed to the proof of the theorem, showing that Λ is a constrained Willmore surface if and only if there exists a holomorphic quadratic differential q satisfying equation (5.4), under the correspondence mentioned above.

First of all, fix a never-zero section σ of Λ and a holomorphic chart z of M. Independently of the choice of such a σ and such a z, set

$$\Lambda^{1,0} := \langle\sigma,\sigma_z\rangle$$

and

$$\Lambda^{0,1} := \langle\sigma,\sigma_{\bar{z}}\rangle,$$

defining in this way two isotropic complex rank 2 subbundles of S, complex conjugate of each other. In view of the non-degeneracy of S, given $i\neq j\in\{0,1\}$, $\operatorname{rank} S = \operatorname{rank}\Lambda^{i,j} + \operatorname{rank}(\Lambda^{i,j})^{\perp}$, showing that the isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ establishes, furthermore, their maximal isotropy in S,

$$(\Lambda^{1,0})^{\perp}\cap S = \Lambda^{1,0},\ \ (\Lambda^{0,1})^{\perp}\cap S = \Lambda^{0,1}.$$

Note that

$$\Lambda = \Lambda^{1,0} \cap \Lambda^{0,1}.$$

Clearly,

$$\Lambda^{1,0} \wedge \Lambda^{1,0} = \langle \sigma \wedge \sigma_z \rangle = \Lambda \wedge \Lambda^{1,0},$$

as well as

$$\Lambda^{0,1} \wedge \Lambda^{0,1} = \langle \sigma \wedge \sigma_{\bar{z}} \rangle = \Lambda \wedge \Lambda^{0,1}.$$

The orthogonality relations of the frame $\sigma, \sigma_z, \sigma_{\bar{z}}$ of

$$\Lambda^{(1)} = \langle \sigma, \sigma_z, \sigma_{\bar{z}} \rangle$$

show that $\sigma \wedge \sigma_z$ and $\sigma \wedge \sigma_{\bar{z}}$ are linearly independent and, therefore, that

$$\Lambda \wedge \Lambda^{(1)} = \Lambda \wedge \Lambda^{1,0} \oplus \Lambda \wedge \Lambda^{0,1}.$$

Remark 5.8. *Given ξ a section of $\Lambda \wedge \Lambda^{1,0}$ (respectively, $\Lambda \wedge \Lambda^{0,1}$), ξ vanishes everywhere outside $\langle \sigma_{\bar{z}}, \sigma_{z\bar{z}} \rangle$ (respectively, $\langle \sigma_z, \sigma_{z\bar{z}} \rangle$) and*

$$\xi \sigma_{\bar{z}} = \lambda \sigma, \quad \xi \sigma_{z\bar{z}} = \lambda \sigma_z,$$

(respectively, $\xi \sigma_z = \lambda \sigma$, $\xi \sigma_{z\bar{z}} = \lambda \sigma_{\bar{z}}$), for some $\lambda \in C^\infty(M, \mathbb{R})$. In particular, ξ is determined by $\xi \sigma_{\bar{z}}$ (respectively, $\xi \sigma_z$), or, equivalently, by $\xi \sigma_{z\bar{z}}$, or, yet again, equivalently, by ξu, fixing $u \in S \backslash \Lambda^\perp$.

Observe that

$$\mathcal{D}^{1,0}\, \Gamma(\Lambda^{1,0}) \subset \Omega^{1,0}(\Lambda^{1,0}), \quad \mathcal{D}^{0,1}\, \Gamma(\Lambda^{0,1}) \subset \Omega^{0,1}(\Lambda^{0,1}). \tag{5.7}$$

Indeed, given $\lambda, \mu \in \Gamma(\underline{\mathbb{C}})$, $\mathcal{D}_{\delta_z}(\lambda \sigma^z + \mu \sigma^z_z) = \lambda_z\, \sigma^z + (\lambda + \mu_z)\, \sigma^z_z + \mu\, \pi_S(\sigma^z_{zz}) \in \Gamma(\Lambda^{1,0})$ and, similarly, we verify that $\mathcal{D}_{\delta_{\bar{z}}}(\lambda \sigma^{\bar{z}} + \mu \sigma^{\bar{z}}_{\bar{z}}) \in \Gamma(\Lambda^{0,1})$. Following (5.7), we get

$$\mathcal{D}^{1,0}\, \Gamma(\Lambda \wedge \Lambda^{1,0}) \subset \Omega^{1,0}(\Lambda \wedge \Lambda^{1,0}), \quad \mathcal{D}^{0,1}\, \Gamma(\Lambda \wedge \Lambda^{0,1}) \subset \Omega^{0,1}(\Lambda \wedge \Lambda^{0,1}). \tag{5.8}$$

Observe, on the other hand, that, as $\sigma_{\bar{z}}$ is a section of S, $\mathcal{N}_{\delta_z} \sigma_{\bar{z}} = \pi_{S^\perp}(d_{\delta_z} \sigma_{\bar{z}}) = 0$, and, therefore, in view of (2.27), $\mathcal{N}^{1,0} \Lambda^{0,1} = 0$, or, equivalently, $\mathcal{N}^{0,1} \Lambda^{0,1} = 0$,

$$\mathcal{N}^{1,0} \Lambda^{0,1} = 0 = \mathcal{N}^{0,1} \Lambda^{0,1}. \tag{5.9}$$

It is opportune to observe that, in view of the skew-symmetry of $\mathcal{N}$ and of the maximal isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ in S, it follows, in particular,

$$\mathcal{N}^{1,0} S^\perp \subset \Lambda^{0,1}, \;\; \mathcal{N}^{0,1} S^\perp \subset \Lambda^{1,0}. \tag{5.10}$$

Now recall that a quadratic differential is a 2-tensor represented locally, in the domain of z, as $Q := f^z dz^2 + \overline{f^z} d\bar{z}^2$, with $f^z \in C^\infty(M, \mathbb{C})$. We may refer to Q as *the quadratic differential defined locally by $f^z dz^2$*. The transformation rule for f^z under change of holomorphic coordinates is

$$f^\omega = \omega_z^{-2}\, f^z, \tag{5.11}$$

given another holomorphic chart ω. Q is said to be holomorphic if f^z is holomorphic.[4] We denote the vector space of holomorphic quadratic differentials on M by $H^0(K)$. It is well-known that the vector space of holomorphic quadratic differentials on a compact Riemann surface is finite dimensional (see, for example, [**34**]).

Given a 1-form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, we define a quadratic differential by

$$q_Q := q^z dz^2 + \overline{q^z} d\bar{z}^2,$$

for $q^z \in C^\infty(M, \mathbb{C})$ defined by[5]

$$q^z \sigma := -2\, q_{\delta_z} \sigma_z. \tag{5.12}$$

We shall verify that this is, indeed, well-defined. The independence of (5.12) with respect to σ is a consequence of the fact that $q\Lambda = 0$. On the other hand, the fact that σ_z is a section of $\Lambda^\perp$ ensures that $q_{\delta_z}\sigma_z \in \Gamma(\langle \sigma \rangle)$, determining $q^z \in \Gamma(\underline{\mathbb{C}})$.

A simple, yet crucial, observation is that, in view of Remark 5.8, in the case $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$, q^z is, equivalently, defined by

$$q^z \tau = -2\, q_{\delta_z}(\mathcal{D}_{\delta_z} \tau), \tag{5.13}$$

for every $\tau \in \Gamma(\Lambda^{0,1})$.

Conversely, a quadratic differential $Q = f^z dz^2 + \overline{f^z} d\bar{z}^2$ determines a real 1-form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$ and $Q = q_Q$ by setting $q_{\delta_z}\sigma_z := -\frac{1}{2} f^z \sigma$. We have established a $1-1$ correspondence

$$q \leftrightarrow q_Q \tag{5.14}$$

between real forms $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$ and quadratic differentials.

Lemma 5.9. *Suppose $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ is real and $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$. In that case, the quadratic differential q_Q is holomorphic if and only if $d^{\mathcal{D}} q = 0$.*

Proof. In view of equation (1.9), the 2-form $d^{\mathcal{D}} q$ vanishes if and only if

$$d^{\mathcal{D}} q\,(\delta_z, \delta_{\bar{z}}) = \mathcal{D}_{\delta_z} \circ q_{\delta_{\bar{z}}} - q_{\delta_{\bar{z}}} \circ \mathcal{D}_{\delta_z} - \mathcal{D}_{\delta_{\bar{z}}} \circ q_{\delta_z} + q_{\delta_z} \circ \mathcal{D}_{\delta_{\bar{z}}}$$

does. As q is real and $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$, $q^{0,1} = \overline{q^{1,0}} \in \Omega^{0,1}(\Lambda \wedge \Lambda^{1,0})$ and, therefore,

$$q^{1,0}\Lambda^{0,1} = 0 = q^{0,1}\Lambda^{1,0};$$

and, on the other hand,

$$qS^\perp = 0.$$

[4]Noting that, given another holomorphic chart ω, $f^\omega_{\bar{z}} = \omega_z^{-2} f^z_{\bar{z}}$, so that $f^\omega_{\bar{z}}$ vanishes if and only if so does $f^z_{\bar{z}}$.

[5]Scaling $q_{\delta_z}\sigma_z$ by -2 will avoid some extra scaling in future equations.

In particular, as $S^\perp$ is $\mathcal{D}$-parallel,

$$d^{\mathcal{D}}q\,(\delta_z,\delta_{\bar z})\,\sigma^z = 0 = d^{\mathcal{D}}q\,(\delta_z,\delta_{\bar z})\,s^\perp,$$

for all $s^\perp \in \Gamma(S^\perp)$. On the other hand, having in consideration that $\pi_S(\sigma^z_{zz}) \in \Gamma(\Lambda)$ and according to equation (5.13),

$$d^{\mathcal{D}}q\,(\delta_z,\delta_{\bar z})\,\sigma^z_z = -\mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_z) + q_{\delta_z}\sigma^z_{z\bar z} = \frac{1}{2}\,\mathcal{D}_{\delta_{\bar z}}(q^z\sigma^z) - \frac{1}{2}\,q^z\sigma^z_{\bar z} = \frac{1}{2}\,q^z_{\bar z}\sigma^z.$$

We conclude that q_Q is holomorphic, $q^z_{\bar z} = 0$, if and only if $d^{\mathcal{D}}q\,(\delta_z,\delta_{\bar z})\,\sigma^z_z$ vanishes. In its turn, by the reality of q (and that of $\mathcal{D}$ and of σ^z),

$$d^{\mathcal{D}}q(\delta_z,\delta_{\bar z})\,\sigma^z_{\bar z} = -\overline{d^{\mathcal{D}}q(\delta_z,\delta_{\bar z})\,\sigma^z_z}.$$

Lastly, we contemplate

$$d^{\mathcal{D}}q(\delta_z,\delta_{\bar z})\,\sigma^z_{z\bar z} = \overline{\mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_{z\bar z})} - q_{\delta_{\bar z}}(\pi_S(\sigma^z_{zz\bar z})) - \mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_{z\bar z}) + \overline{q_{\delta_{\bar z}}(\pi_S(\sigma^z_{zz\bar z}))}.$$

First of all,

$$\pi_S(\sigma^z_{zz\bar z}) = \pi_S(-\frac{1}{2}\,c^z\sigma^z + k^z)_{\bar z} = -\frac{1}{2}\,c^z\sigma^z_{\bar z} - \frac{1}{2}\,c^z_{\bar z}\,\sigma^z + \pi_S(k^z_{\bar z}),$$

so that

$$q_{\delta_{\bar z}}(\pi_S(\sigma^z_{zz\bar z})) = -\frac{1}{2}\,c^z q_{\delta_{\bar z}}\sigma^z_{\bar z} + q_{\delta_{\bar z}}k^z_{\bar z} = \frac{1}{4}\,c^z\,\overline{q^z}\,\sigma^z + q_{\delta_{\bar z}}k^z_{\bar z}.$$

Together with $(k^z,\sigma^z_{\bar z}) = 0 = (k^z,\sigma^z_{z\bar z})$, differentiation of $(k^z,\sigma^z) = 0 = (k^z,\sigma^z_z)$ shows that $(k^z_{\bar z},\sigma^z) = 0 = (k^z_{\bar z},\sigma^z_z)$, or, equivalently, that

$$\pi_S k^z_{\bar z} \in \Gamma(\Lambda^{1,0}). \tag{5.15}$$

Hence $q_{\delta_{\bar z}}k^z_{\bar z} = 0$. On the other hand,

$$\mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_{z\bar z}) = -\frac{1}{2}\,q^z\pi_S(\sigma^z_{\bar z\bar z}) - \frac{1}{2}\,q^z_{\bar z}\,\sigma^z_{\bar z} = \frac{1}{4}\,\overline{c^z}\,q^z\sigma^z - \frac{1}{2}\,q^z_{\bar z}\,\sigma^z_{\bar z}.$$

We conclude that

$$d^{\mathcal{D}}q(\delta_z,\delta_{\bar z})\,\sigma^z_{z\bar z} = -\frac{1}{2}\,\overline{q^z_{\bar z}}\,\sigma^z_z + \frac{1}{2}\,q^z_{\bar z}\,\sigma^z_{\bar z}$$

vanishes if and only if $q^z_{\bar z}$ does, which completes the proof. □

The next result establishes, in particular, that if q is a multiplier to Λ then $q^{1,0}$ takes values in $\Lambda\wedge\Lambda^{0,1}$.

Lemma 5.10. *Given $q \in \Omega^1(\Lambda\wedge\Lambda^{(1)})$ real,*

i) if $d^{\mathcal{D}}q = 0$ then $q^{1,0} \in \Omega^{1,0}(\Lambda\wedge\Lambda^{0,1})$ or, equivalently, $q^{0,1} \in \Omega^{0,1}(\Lambda\wedge\Lambda^{1,0})$;

ii) $d^{\mathcal{D}}q = 0$ if and only if $d^{\mathcal{D}}q^{1,0} = 0$, or, equivalently, $d^{\mathcal{D}}q^{0,1} = 0$;

iii) $d^{\mathcal{D}}q = 0$ if and only if $d^{\mathcal{D}} * q = 0$.

Before we proceed to the proof of the lemma, observe that a section ξ of $\Lambda\wedge\Lambda^{(1)}$ is a section of $\Lambda\wedge\Lambda^{1,0}$, or, equivalently, $\overline{\xi}$ is a section of $\Lambda\wedge\Lambda^{0,1}$, if and only if $\xi(\sigma_z) = 0$.

Now we proceed to the proof of Lemma 5.10.

Proof. To prove i), we prove, equivalently, that, if $d^{\mathcal{D}}q = 0$, then $q_{\delta_{\bar z}}\sigma^z_z = 0$. First, if $d^{\mathcal{D}}q = 0$, then, in particular, $d^{\mathcal{D}}q\,(\delta_z, \delta_{\bar z})\,\sigma^z = 0$, or, equivalently,

$$\mathcal{D}_{\delta_z}(q_{\delta_{\bar z}}\sigma^z) - q_{\delta_{\bar z}}(\mathcal{D}_{\delta_z}\sigma^z) - \mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z) + q_{\delta_z}(\mathcal{D}_{\delta_{\bar z}}\sigma^z) = 0,$$

establishing

$$q_{\delta_z}\sigma^z_{\bar z} = q_{\delta_{\bar z}}\sigma^z_z. \tag{5.16}$$

In its turn, $d^{\mathcal{D}}q\,(\delta_z, \delta_{\bar z})\,\sigma^z_z = 0$ implies

$$\mathcal{D}_{\delta_z}(q_{\delta_{\bar z}}\sigma^z_z) - q_{\delta_{\bar z}}(\pi_S(\sigma^z_{zz})) - \mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_z) + q_{\delta_z}\sigma^z_{z\bar z} = 0,$$

or, equivalently,

$$\mathcal{D}_{\delta_z}(q_{\delta_{\bar z}}\sigma^z_z) - \mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_z) + q_{\delta_z}\sigma^z_{z\bar z} = 0.$$

On the other hand, the orthogonality relations between σ^z, σ^z_z, $\sigma^z_{\bar z}$ and $\sigma^z_{z\bar z}$ show that

$$q\sigma^z_{z\bar z} = \mu\sigma_z + \eta\sigma^z_{\bar z}, \tag{5.17}$$

for some $\mu, \eta \in \Omega^1(\underline{\mathbb{C}})$. Hence

$$\mathcal{D}_{\delta_z}(q_{\delta_{\bar z}}\sigma^z_z) + \mu_{\delta_z}\sigma^z_z = \mathcal{D}_{\delta_{\bar z}}(q_{\delta_z}\sigma^z_z) - \eta_{\delta_z}\sigma^z_{\bar z}.$$

It is obvious that a section of $\Lambda\wedge\Lambda^{(1)}$ transforms sections of $\Lambda^{(1)}$ into sections of Λ, so that, in particular, both $q_{\delta_{\bar z}}\sigma^z_z$ and $q_{\delta_z}\sigma^z_z$ are sections of Λ. By (5.7), we conclude that $\mathcal{D}_{\delta_z}(q_{\delta_{\bar z}}\sigma^z_z) + \mu_{\delta_z}\sigma^z_z$ is a section of $\Lambda^{1,0}\cap\Lambda^{0,1} = \Lambda$. Write

$$q_{\delta_{\bar z}}\sigma^z_z = \lambda\sigma^z, \tag{5.18}$$

with $\lambda \in \Gamma(\underline{\mathbb{C}})$. Then $\lambda_z\sigma^z + (\lambda + \mu_{\delta_z})\sigma^z_z = \gamma\sigma^z$, for some $\gamma \in \Gamma(\underline{\mathbb{C}})$. In particular, $\lambda = -\mu_{\delta_z}$. Equation (5.17) shows, on the other hand, that $q = -2\mu\,\sigma^z\wedge\sigma^z_z - 2\eta\,\sigma^z\wedge\sigma^z_{\bar z}$ and, consequently, equations (5.16) and (5.18) together give $\lambda = \mu_{\delta_z}$, completing the proof of i).

Next we prove ii). If $d^{\mathcal{D}}q$ vanishes, then, following i), and in view of (5.8),

$$d^{\mathcal{D}}q^{1,0} \in \Omega^2(\Lambda\wedge\Lambda^{0,1}), \quad d^{\mathcal{D}}q^{0,1} \in \Omega^2(\Lambda\wedge\Lambda^{1,0}),$$

which, as $\Lambda\wedge\Lambda^{1,0}$ and $\Lambda\wedge\Lambda^{0,1}$ are complementary in $\Lambda\wedge\Lambda^{(1)}$, forces $d^{\mathcal{D}}q^{1,0}$ and $d^{\mathcal{D}}q^{0,1}$ to vanish separately, $d^{\mathcal{D}}q^{1,0} = 0 = d^{\mathcal{D}}q^{0,1}$. On the other hand, as q is real,

$$d^{\mathcal{D}}q^{0,1}(\delta_z, \delta_{\bar z}) = -\overline{d^{\mathcal{D}}q^{1,0}(\delta_z, \delta_{\bar z})},$$

so that $d^{\mathcal{D}}q^{1,0}$ vanishes if and only if $d^{\mathcal{D}}q^{0,1}$ does. In particular, if $d^{\mathcal{D}}q^{1,0}$ vanishes, then, obviously, so does $d^{\mathcal{D}}q = d^{\mathcal{D}}q^{1,0} + d^{\mathcal{D}}q^{0,1}$.

As for iii), it is immediate from ii), as $*q = -iq^{1,0} + iq^{0,1} \in \Omega^1(\Lambda\wedge\Lambda^{(1)})$ is real, as well as q. $\square$

The $1-1$ correspondence given by (5.14) establishes, in particular, a correspondence between holomorphic quadratic differentials and real forms $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with $d^{\mathcal{D}}q = 0$.

Lastly, we proceed to the proof of Theorem 5.6.

Proof. Let $(\Lambda_t)_t$ be a variation of Λ through immersions of M into the projectivized light-cone and $\dot{\Lambda} \in \Gamma(\mathrm{Hom}(\Lambda, \Lambda^\perp/\Lambda))$ be the corresponding variational vector field, cf. (4.20). The variation $(\Lambda_t)_t$ is infinitesimally conformal if and only if

$$\frac{d}{dt}_{|t=0}(d(\Lambda_t)(\delta_{\bar{z}}), d(\Lambda_t)(\delta_{\bar{z}})) = 0,$$

or, equivalently,

$$(\dot{\Lambda}_{\bar{z}}, \Lambda_{\bar{z}}) = 0,$$

noting that $\frac{d}{dt}_{|t=0}$ and $d_{\delta_{\bar{z}}}$ commute, as z is independent of t. Write

$$\dot{\Lambda} = d\Lambda(X) + \nu$$

with $X \in \Gamma(TM)$ and $\nu \in \Gamma(\mathrm{Hom}(\Lambda, S^\perp))$ as defined in Section 4.5. Let π_{T_Λ} and π_{N_Λ} denote the orthogonal projections of $\Lambda^*T\mathbb{P}(\mathcal{L}) = T_\Lambda \oplus N_\Lambda$ onto T_Λ and N_Λ, respectively. By equation (2.1),

$$\dot{\Lambda}_{\bar{z}} = d\Lambda(\bar{\partial}_{\delta_{\bar{z}}}X) + \pi_{N_\Lambda}(d\Lambda(X))_{\bar{z}} + \pi_{T_\Lambda}(\nu_{\bar{z}}) + \pi_{N_\Lambda}(\nu_{\bar{z}}),$$

and, therefore, $(\dot{\Lambda}_{\bar{z}}, \Lambda_{\bar{z}}) = (d\Lambda(\bar{\partial}_{\delta_{\bar{z}}}X), \Lambda_{\bar{z}}) - (A^\nu(\delta_{\bar{z}}), \Lambda_{\bar{z}})$. Let $X^{1,0}$ be the projection of X onto $T^{1,0}M$. By the isotropy of $T^{0,1}M$ and in view of (1.7), it follows that $(\Lambda_t)_t$ is infinitesimally conformal if and only if

$$(d\Lambda(\bar{\partial}_{\delta_{\bar{z}}}X^{1,0}) - A_0^\nu(\delta_{\bar{z}}), \Lambda_{\bar{z}}) = 0, \tag{5.19}$$

for $A_0^\nu := A^\nu - H^\nu I$, the trace-free part of the shape operator A^ν. Equation (2.2) shows that, as the second fundamental form is symmetric, so is the shape operator,

$$(A^\nu X, Y) = (X, A^\nu Y),$$

for all $X, Y \in \Gamma(TM)$; and, therefore, so is as well the trace-free part of the shape operator. It follows that

$$A_0^\nu J = -JA_0^\nu,$$

showing that A_0^ν intertwines the eigenspaces of J. In view of the isotropy of $T^{1,0}M$, we conclude that equation (5.19) holds if and only if $(d\Lambda(\bar{\partial}_{\delta_{\bar{z}}}X^{1,0}) - A_0^\nu(\delta_{\bar{z}}), d\Lambda(Y)) = 0$, for all $Y \in \Gamma(TM)$, or, equivalently,

$$d\Lambda(\bar{\partial}X^{1,0}) = (A_0^\nu)^{0,1}.$$

According to (5.10), $\mathcal{N}^{0,1}\nu \in \Omega^{0,1}(\mathrm{Hom}(\Lambda, \Lambda^{1,0})) = \Omega^{0,1}(\mathrm{Hom}(\Lambda, d\Lambda(T^{1,0}M)/\Lambda))$. Thus $(\mathcal{N}_{\delta_{\bar{z}}}\nu, \Lambda_{\bar{z}}) = (-A_0^\nu(\delta_{\bar{z}}), \Lambda_{\bar{z}})$. We conclude that infinitesimally conformal variations

through immersions are characterized by the equation

$$d\Lambda(\bar{\partial}X^{1,0}) = -\mathcal{N}^{0,1}\nu;$$

or, equivalently,

$$\mathcal{A}\nu \in \Omega^2(\mathrm{Im}\,\bar{\partial}),$$

for $\mathcal{A} := -d\Lambda^{-1} \circ \mathcal{N}^{0,1}$.

Having said so, recall equation (4.23), establishing

$$\dot{W} = \ll d^{\mathcal{D}} * \mathcal{N}, \dot{\Lambda} \gg = \ll d^{\mathcal{D}} * \mathcal{N}, \nu \gg$$

for $\ll, \gg$ the non-degenerate pairing between 2-forms over M and normal variations defined in [**6**]. We conclude that Λ is constrained Willmore if and only $d^{\mathcal{D}} * \mathcal{N} \perp_{\ll,\gg} \{\nu : \mathcal{A}\nu \in \Omega^2(\mathrm{Im}\,\bar{\partial})\}$.[6] The reference to the pairing shall be omitted from now on.

From this point, the proof of the theorem consists of a straightforward generalization to n-space of the argument presented in [**7**] for the particular case of $n = 3$. Basically, in [**7**], it is presented a pairing between the space of quadratic differentials and the space of J-anti-commuting endomorphisms of TM, with respect to which, by Weyl's lemma,

$$\mathrm{Im}\,\bar{\partial} = (H^0 K)^{\perp}. \tag{5.20}$$

In view of equation (5.20), Λ is constrained Willmore if and only if $\nu \in (\mathcal{A}^* H^0(K))^{\perp}$, or, equivalently,

$$d^{\mathcal{D}} * \mathcal{N} \in \Omega^2((\mathcal{A}^* H^0(K))^{\perp\perp}).$$

As M is compact, $H^0(K)$ is finite dimensional and, therefore,

$$(\mathcal{A}^* H^0(K))^{\perp\perp} = \mathcal{A}^* H^0(K).$$

We conclude that the surface Λ is constrained Willmore if and only if there exists some homomorphic quadratic differential q such that $d^{\mathcal{D}} * \mathcal{N} = \mathcal{A}^* q$. The final conclusion follows then, after some computation involving the pairings mentioned above. $\square$

5.4. A constrained Willmore surface equation on the Hopf differential and the Schwarzian derivative

Theorem 5.6 consists of a reformulation of the characterization of constrained Willmore surfaces in space-forms established in [**14**], which we deduce in this section.

Let $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ be real with $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$. According to Remark 5.7, equation (5.4) holds if and only if, fixing a holomorphic chart z of M,

$$d^{\mathcal{D}} * \mathcal{N}(\delta_z, \delta_{\bar{z}})\,\sigma^z_{z\bar{z}} = 2[q \wedge *\mathcal{N}](\delta_z, \delta_{\bar{z}})\,\sigma^z_{z\bar{z}},$$

[6]The Willmore surface equation follows immediately as the particular case where no constraint is considered.

or, equivalently,

$$(\mathcal{D}_{\delta_z}\circ\mathcal{N}_{\delta_{\bar z}}-\mathcal{N}_{\delta_{\bar z}}\circ\mathcal{D}_{\delta_z}+\overline{\mathcal{D}_{\delta_z}\circ\mathcal{N}_{\delta_{\bar z}}-\mathcal{N}_{\delta_{\bar z}}\circ\mathcal{D}_{\delta_z}})\,\sigma^z_{z\bar z}=2[q_{\delta_z},\mathcal{N}_{\delta_{\bar z}}]\sigma^z_{z\bar z}+2\overline{[q_{\delta_z},\mathcal{N}_{\delta_{\bar z}}]\sigma^z_{z\bar z}}.$$

On the one hand,

$$\mathcal{D}_{\delta_z}\circ\mathcal{N}_{\delta_{\bar z}}\sigma^z_{z\bar z}=\mathcal{D}_{\delta_z}\circ\pi_{S^\perp}\overline{(\sigma^z_{zz})_{\bar z}}=\mathcal{D}_{\delta_z}\circ\pi_{S^\perp}(\,\overline{k^z_{\bar z}}\,)=\overline{\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z}.$$

On the other hand,

$$\mathcal{N}_{\delta_{\bar z}}\circ\mathcal{D}_{\delta_z}\sigma^z_{z\bar z}=\mathcal{N}_{\delta_{\bar z}}\circ\pi_S(\sigma^z_{zz})_{\bar z}=-\frac{1}{2}\,c^z\mathcal{N}_{\delta_{\bar z}}\sigma^z_{\bar z}+\mathcal{N}_{\delta_{\bar z}}\circ\pi_S k^z_{\bar z}.$$

Note that, as $\sigma^z_{\bar z}$ and $\sigma^z_{z\bar z}$ are sections of S, $\mathcal{N}_{\delta_{\bar z}}\sigma^z_{\bar z}=\pi_{S^\perp}(d_{\delta_{\bar z}}\sigma^z_{\bar z})=0$. Together with (2.27), this establishes $\mathcal{N}^{0,1}\Lambda^{1,0}=0$. By (5.15), it follows that $\mathcal{N}_{\delta_{\bar z}}\circ\mathcal{D}_{\delta_z}\sigma^z_{z\bar z}=-\frac{1}{2}\,c^z\overline{k^z}$. Lastly,

$$[q_{\delta_z},\mathcal{N}_{\delta_{\bar z}}]\sigma^z_{z\bar z}=-\mathcal{N}_{\delta_{\bar z}}q_{\delta_z}\sigma^z_{z\bar z}=\frac{1}{2}\,q^z\mathcal{N}_{\delta_{\bar z}}\sigma^z_{\bar z}=\frac{1}{2}\,q^z\,\overline{\pi_{S^\perp}\sigma^z_{zz}}=\frac{1}{2}\,q^z\,\overline{k^z}.$$

We conclude that equation (5.4) holds if and only if

$$\mathrm{Re}(\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{1}{2}\,\overline{c^z}k^z)=\mathrm{Re}(\overline{q^z}\,k^z).$$

Equation (33b) in [**14**], establishing the reality of $\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{1}{2}\,\overline{c^z}k^z$, leads us to the following constrained Willmore surface equation in terms of the Hopf differential and the Schwarzian derivative, presented in [**14**]:[7]

Lemma 5.11. *Let q be a real 1-form with values in $\Lambda\wedge\Lambda^{(1)}$ with $q^{1,0}\in\Omega^{1,0}(\Lambda\wedge\Lambda^{0,1})$. Λ is a q-constrained Willmore surface if and only if, fixing a holomorphic chart z of M, q^z is holomorphic and*

$$\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{\overline{c^z}}{2}\,k^z=\mathrm{Re}\,(\overline{q^z}\,k^z). \tag{5.21}$$

The surface Λ is constrained Willmore if and only there exists a holomorphic map $q^z\in C^\infty(M,\mathbb{C})$ satisfying equation (5.21), in which case, the quadratic differential defined locally by $q^z dz^2$ is, under the correspondence given by (5.14), a multiplier to Λ.

[7] The argument above shows that the constrained Willmore surface equation on a holomorphic quadratic differential, presented in [**14**] with no reference to a multiplier, is equivalent to equation (5.4), under the correspondence given by (5.14).

CHAPTER 6

Constrained Willmore surfaces: spectral deformation and Bäcklund transformation

In [**11**], F. Burstall and D. Calderbank present a characterization of constrained Willmore surfaces in spherical space in terms of the flatness of a certain loop of metric connections. In view of this characterization, we characterize constrained Willmore surfaces in space-forms in terms of the *constrained harmonicity* of the central sphere congruence, generalizing the characterization of Willmore surfaces in space-forms in terms of the harmonicity of the central sphere congruence. This will enable us to define a spectral deformation of constrained Willmore surfaces in space-forms, by the action of the loop of flat metric connections above, as well as a *Bäcklund transformation*, by applying a dressing action. Both Bäcklund transformation and spectral deformation corresponding to the zero multiplier preserve the class of Willmore surfaces. We establish a permutability between spectral deformation and Bäcklund transformation of constrained Willmore surfaces in space-forms.

Let $\tilde{d}$ be a flat metric connection on $\underline{\mathbb{C}}^{n+2}$. It is obvious, but, nevertheless, opportune to remark that $\tilde{d}$ defines a connection on $\underline{\mathbb{R}}^{n+1,1}$, i.e., $\tilde{d}$ preserves the reality of sections of $\underline{\mathbb{C}}^{n+2} = (\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$, if and only if $\tilde{d}$ is real,

$$\overline{\tilde{d}} = \tilde{d}.$$

Equally basic is to remark that, if $\hat{d}_1$ and $\hat{d}_2$ are real connections on $\underline{\mathbb{C}}^{n+2}$, then an isomorphism $\phi : (\underline{\mathbb{C}}^{n+2}, \hat{d}_1) \to (\underline{\mathbb{C}}^{n+2}, \hat{d}_2)$ is real,

$$\overline{\phi} = \phi,$$

if and only if it defines an isomorphism $\phi : (\underline{\mathbb{R}}^{n+1,}, \hat{d}_1) \to (\underline{\mathbb{R}}^{n+1,1}, \hat{d}_2)$. Note also that the real bundles in $\underline{\mathbb{C}}^{n+2}$ - those preserved by complex conjugation - are the complexifications of bundles in $\underline{\mathbb{R}}^{n+1,1}$: given $W \subset \underline{\mathbb{C}}^{n+2}$ real,

$$W = W \cap \underline{\mathbb{R}}^{n+1,1} \oplus W \cap i\underline{\mathbb{R}}^{n+1,1} = (W \cap \underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}.$$

Throughout this chapter, let V be a non-degenerate subbundle of $\underline{\mathbb{C}}^{n+2}$,

$$\underline{\mathbb{C}}^{n+2} = V \oplus V^{\perp},$$

and π_V and $\pi_{V^\perp}$ denote the orthogonal projections of $\underline{\mathbb{C}}^{n+2}$ onto V and $V^\perp$, respectively.

Fix a conformal structure $\mathcal{C}$ in M.

6.1. Constrained harmonicity of bundles

A multiplier to a surface Λ in the projectivized light-cone is, in particular, a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. For such a q, equations (5.3) and (5.4), together, encode the flatness of the connection $d_q^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2}-1)\,q^{1,0} + (\lambda^2-1)\,q^{0,1}$, on $(\underline{\mathbb{R}}^{n+1,1})^{\mathbb{C}}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$, or, equivalently, for all $\lambda \in S^1$. Constrained Willmore surfaces in space-forms, admitting q as a multiplier, are characterized by the flatness of the S^1-family of metric connections d_q^λ on $\underline{\mathbb{R}}^{n+1,1}$, in an integrable systems interpretation due to F. Burstall and D. Calderbank [**11**]. This characterization will enable us to define a spectral deformation of constrained Willmore surfaces in space-forms, by the action of the loop of flat metric connections d_q^λ, as well as a *Bäcklund transformation*, by applying a dressing action. Our transformations of constrained Willmore surfaces will be based on the *constrained harmonicity* of the central sphere congruence. Given $\hat{d}$ a flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and V a non-degenerate subbundle of $\underline{\mathbb{C}}^{n+2}$, we generalize naturally the decomposition (2.26) to a decomposition $\hat{d} = \hat{\mathcal{D}}_V + \hat{\mathcal{N}}_V$ and, given $q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$, define then, for each $\lambda \in \mathbb{C}\backslash\{0\}$, a connection $\hat{d}_V^{\lambda,q} := \hat{\mathcal{D}}_V + \lambda^{-1}\hat{\mathcal{N}}_V^{1,0} + \lambda\hat{\mathcal{N}}_V^{0,1} + (\lambda^{-2}-1)q^{1,0} + (\lambda^2-1)q^{0,1}$, on $\underline{\mathbb{C}}^{n+2}$, generalizing $d_q^\lambda = d_S^{\lambda,q}$. We define V to be $(q,\hat{d})$-*constrained harmonic* if $\hat{d}_V^{\lambda,q}$ is flat, for all $\lambda \in \mathbb{C}\backslash\{0\}$, or, equivalently, for all $\lambda \in S^1$. A simple, yet crucial, observation is that, given $\tilde{d}$ another flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and $\phi : (\underline{\mathbb{C}}^{n+2}, \tilde{d}) \to (\underline{\mathbb{C}}^{n+2}, \hat{d})$ an isomorphism of bundles provided with a metric and a connection, V is $(q,\tilde{d})$-constrained harmonic if and only if ϕV is $(\mathrm{Ad}_\phi q, \hat{d})$-constrained harmonic. The constrained harmonicity of a bundle applies to the central sphere congruence, providing a characterization of constrained Willmore surfaces in space-forms.

Consider the decomposition

$$\tilde{d} = \mathcal{D}_V^{\tilde{d}} + \mathcal{N}_V^{\tilde{d}}$$

for $\mathcal{D}_V^{\tilde{d}}$ the connection on $\underline{\mathbb{C}}^{n+2}$ defined by

$$\mathcal{D}_V^{\tilde{d}} := \pi_V \circ \tilde{d} \circ \pi_V + \pi_{V^\perp} \circ \tilde{d} \circ \pi_{V^\perp}.$$

Note that, as $\tilde{d}$ is a metric connection, so is $\mathcal{D}_V^{\tilde{d}}$: given $\eta, \mu \in \Gamma(\underline{\mathbb{C}}^{n+2})$,

$$\begin{aligned}
d(\eta,\mu) &= d(\pi_V\eta, \pi_V\mu) + d(\pi_{V^\perp}\eta, \pi_{V^\perp}\mu)\\
&= (\tilde{d}(\pi_V\eta), \pi_V\mu) + (\pi_V\eta, \tilde{d}(\pi_V\mu)) + (\tilde{d}(\pi_{V^\perp}\eta), \pi_{V^\perp}\mu) + (\pi_{V^\perp}\eta, \tilde{d}(\pi_{V^\perp}\mu))\\
&= (\mathcal{D}_V^{\tilde{d}}\eta, \mu) + (\eta, \mathcal{D}_V^{\tilde{d}}\mu).
\end{aligned}$$

Equivalently,

$$\mathcal{N}_V^{\tilde{d}} := d - \mathcal{D}_V^{\tilde{d}} = \pi_{V^\perp} \circ \tilde{d} \circ \pi_V + \pi_V \circ \tilde{d} \circ \pi_{V^\perp}$$

is skew-symmetric,

$$\mathcal{N}_V^{\tilde{d}} \in \Omega^1(V \wedge V^\perp).$$

In the particular case $\tilde{d} = d$, the trivial flat connection, we shall omit the reference to $\tilde{d}$.

Next we present the concept of *constrained harmonicity* of a bundle, which will apply to the central sphere congruence to provide a characterization of constrained Willmore surfaces, in view of the characterization of these surfaces in terms of the flatness of a certain loop of metric connections presented in [**11**], which we shall address later on. In fact, the following definition encodes the characterization of constrained harmonicity for a general bundle in terms of the flatness of a loop of metric connections generalizing the one above, as we shall verify later on.

Let q be a 1-form with values in $\wedge^2 V \oplus \wedge^2 V^\perp \subset o(\underline{\mathbb{C}}^{n+2})$.

Definition 6.1. *V is said to be $(q, \tilde{d})$*-constrained harmonic *if*

i) $d^{\mathcal{D}_V^{\tilde{d}}} q^{1,0} = \frac{1}{2}\,[q \wedge q] = d^{\mathcal{D}_V^{\tilde{d}}} q^{0,1}$;

ii) $d^{\mathcal{D}_V^{\tilde{d}}} * \mathcal{N}_V^{\tilde{d}} = 2\,[q \wedge * \mathcal{N}_V^{\tilde{d}}]$.

By *$\tilde{d}$-constrained harmonicity* of V we mean the existence of a *multiplier* to V with respect to $\tilde{d}$, i.e., a 1-form q with values in $\wedge^2 V \oplus \wedge^2 V^\perp$ for which V is $(q, \tilde{d})$-constrained harmonic. In the particular case of $\tilde{d} = d$, we shall, alternatively, omit the reference to $\tilde{d}$ and refer simply to *constrained harmonicity* or, when specifying $q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$, to *$q$-constrained harmonicity*. We shall refer to $(0, \tilde{d})$-constrained harmonic bundles, alternatively, as *$\tilde{d}$-harmonic bundles*.

It is useful to observe that, as $\mathcal{N}_V^{\tilde{d}}$ and q take values in $V \wedge V^\perp$ and $\wedge^2 V \oplus \wedge^2 V^\perp$, respectively, and V and $V^\perp$ are $\mathcal{D}_V^{\tilde{d}}$-parallel,

$$d^{\mathcal{D}_V^{\tilde{d}}} * \mathcal{N}_V^{\tilde{d}},\ [q \wedge * \mathcal{N}_V^{\tilde{d}}] \in \Omega^2(V \wedge V^\perp), \tag{6.1}$$

whereas

$$d^{\mathcal{D}_V^{\tilde{d}}} q^{1,0}, d^{\mathcal{D}_V^{\tilde{d}}} q^{0,1}, [q \wedge q] \in \Omega^2(\wedge^2 V \oplus \wedge^2 V^\perp). \tag{6.2}$$

The same argument establishes both $R^{\mathcal{D}_V^{\tilde{d}}}$ and $[\mathcal{N}_V^{\tilde{d}} \wedge \mathcal{N}_V^{\tilde{d}}]$ as 2-forms with values in $\Omega^2(\wedge^2 V \oplus \wedge^2 V^\perp)$ and $d^{\mathcal{D}_V^{\tilde{d}}} \mathcal{N}_V^{\tilde{d}}$ as a 2-form with values in $\Omega^2(V \wedge V^\perp)$. We conclude that the flatness of $\tilde{d}$ encodes both

Proposition 6.2. *(Gauss-Ricci equation)*

$$R^{\mathcal{D}_V^{\tilde{d}}} + \frac{1}{2}\,[\mathcal{N}_V^{\tilde{d}} \wedge \mathcal{N}_V^{\tilde{d}}] = 0;$$

and

Proposition 6.3. *(Codazzi equation)*

$$d^{\mathcal{D}_V^{\tilde{d}}}\mathcal{N}_V^{\tilde{d}} = 0.$$

According to the Codazzi equation,

$$d^{\mathcal{D}_V^{\tilde{d}}}(\mathcal{N}_V^{\tilde{d}})^{1,0} = -d^{\mathcal{D}_V^{\tilde{d}}}(\mathcal{N}_V^{\tilde{d}})^{0,1},$$

making clear that:

Proposition 6.4. *The $\tilde{d}$-harmonicity of V is characterized, equivalently, by any of the following equations:*

i) $d^{\mathcal{D}_V^{\tilde{d}}} * \mathcal{N}_V^{\tilde{d}} = 0;$

ii) $d^{\mathcal{D}_V^{\tilde{d}}}(\mathcal{N}_V^{\tilde{d}})^{1,0} = 0;$

iii) $d^{\mathcal{D}_V^{\tilde{d}}}(\mathcal{N}_V^{\tilde{d}})^{0,1} = 0.$

Define, for each $\lambda \in \mathbb{C}\backslash\{0\}$, a metric connection on $\underline{\mathbb{C}}^{n+2}$ by

$$\tilde{d}_V^{\lambda,q} := \mathcal{D}_V^{\tilde{d}} + \lambda^{-1}(\mathcal{N}_V^{\tilde{d}})^{1,0} + \lambda(\mathcal{N}_V^{\tilde{d}})^{0,1} + (\lambda^{-2}-1)q^{1,0} + (\lambda^2-1)q^{0,1}.$$

In the particular case of $q = 0$, we shall omit the reference to q in $\tilde{d}_V^{\lambda,q}$.

In the particular case $\tilde{d} = d$ and $V = S$, the complexification of the central sphere congruence of a surface Λ, we shall, alternatively, omit the reference to S and refer to $\tilde{d}_V^{\lambda,q}$ as d_q^λ,

$$d_q^\lambda := \mathcal{D} + \lambda^{-1}\mathcal{N}^{1,0} + \lambda\mathcal{N}^{0,1} + (\lambda^{-2}-1)q^{1,0} + (\lambda^2-1)q^{0,1}.$$

Of course, the set of connections $d^\lambda q$, with $\lambda \in S^1$, defines a group, using multiplication of parameters to define the group law. According to [**11**], if q is a real form taking values in $\Lambda \wedge \Lambda^{(1)}$, then the flatness of the loop of metric connections d_q^λ characterizes Λ as a q-constrained Willmore surface.

Before proceeding any further, we dedicate a few moments to some useful general remarks about the family of connections $\tilde{d}_V^{\lambda,q}$. First note that

$$\tilde{d}_V^{1,q} = \tilde{d}.$$

Note, on the other hand, that, in view of the $\mathcal{D}_V^{\tilde{d}}$-parallelness of V and $V^\perp$, together with the intertwining of V and $V^\perp$ by $\mathcal{N}_V^{\tilde{d}}$, and the fact that q preserves V and $V^\perp$, we have

$$\mathcal{D}_V^{\tilde{d}_V^{\lambda,q}} = \mathcal{D}_V^{\tilde{d}} + (\lambda^{-2}-1)q^{1,0} + (\lambda^2-1)q^{0,1}, \quad \mathcal{N}_V^{\tilde{d}_V^{\lambda,q}} = \lambda^{-1}(\mathcal{N}_V^{\tilde{d}})^{1,0} + \lambda(\mathcal{N}_V^{\tilde{d}})^{0,1}.$$

Set

$$q_\lambda := \lambda^{-2}q^{1,0} + \lambda^2 q^{0,1}.$$

Then

$$(\tilde{d}_V^{\lambda,q})_V^{\mu,q_\lambda} = \mathcal{D}_V^{\tilde{d}_V^{\lambda,q}} + \mu^{-1}(\mathcal{N}_V^{\tilde{d}_V^{\lambda,q}})^{1,0} + \mu(\mathcal{N}_V^{\tilde{d}_V^{\lambda,q}})^{0,1} + (\mu^{-2}-1)q_\lambda^{1,0} + (\mu^2-1)q_\lambda^{0,1}$$
$$= \mathcal{D}_V^{\tilde{d}} + (\lambda\mu)^{-1}(\mathcal{N}_V^{\tilde{d}})^{1,0} + \lambda\mu(\mathcal{N}_V^{\tilde{d}})^{0,1} + ((\lambda\mu)^{-2}-1)q^{1,0} + ((\lambda\mu)^2-1)q^{0,1}$$

and, ultimately,

$$(\tilde{d}_V^{\lambda,q})_V^{\mu,q_\lambda} = \tilde{d}_V^{\lambda\mu,q}. \tag{6.3}$$

It will also be useful to observe that, given another flat metric connection $\hat{d}$ on $\underline{\mathbb{C}}^{n+2}$ and an isomorphism $\psi : (\underline{\mathbb{C}}^{n+2}, \hat{d}) \to (\underline{\mathbb{C}}^{n+2}, \tilde{d})$ of bundles preserving connections, we have

$$\mathcal{D}_{\psi V}^{\tilde{d}} = \psi \circ \mathcal{D}_V^{\hat{d}} \circ \psi^{-1}, \quad \mathcal{N}_{\psi V}^{\tilde{d}} = \psi\, \mathcal{N}_V^{\hat{d}}\, \psi^{-1} \tag{6.4}$$

and, therefore,

$$\tilde{d}_{\psi V}^{\lambda,q} = \psi \circ \hat{d}_V^{\lambda,\mathrm{Ad}_{\psi^{-1}}q} \circ \psi^{-1}. \tag{6.5}$$

Remark 6.5. *If V is real, then, obviously, so is $V^\perp$, so that, in particular, $\overline{\pi_V} = \pi_V$, $\overline{\pi_{V^\perp}} = \pi_{V^\perp}$ and therefore, if $\tilde{d}$ is real, then $\overline{\mathcal{D}_V^{\tilde{d}}} = \mathcal{D}_V^{\tilde{d}}$, $\overline{\mathcal{N}_V^{\tilde{d}}} = \mathcal{N}_V^{\tilde{d}}$. If V, q and $\tilde{d}$ are real, then so is $\tilde{d}_V^{\lambda,q}$, for all $\lambda \in S^1$.*

The characterization of the harmonicity of V, as a map into a Grassmannian, in terms of the flatness of the S^1-family of metric connections d_V^λ, cf. K. Uhlenbeck [**56**], generalizes to a characterization of $\tilde{d}$-constrained harmonicity, as follows (see also Remark 6.7):

Theorem 6.6. *V is $(q, \tilde{d})$-constrained harmonic if and only if $\tilde{d}_V^{\lambda,q}$ is a flat connection, for each $\lambda \in \mathbb{C}\backslash\{0\}$.*

The proof of the theorem will consist of showing that the $(q, \tilde{d})$-constrained harmonicity of V establishes and, in fact, encodes, in view of the flatness of $\tilde{d}$ (Gauss-Ricci and Codazzi equations), the flatness of $\tilde{d}_V^{\lambda,q}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$.

Proof. For simplicity, write $\tilde{\mathcal{D}}_V$ and $\tilde{\mathcal{N}}_V$ for $\mathcal{D}_V^{\tilde{d}}$ and $\mathcal{N}_V^{\tilde{d}}$. The curvature tensor of $\tilde{d}_V^{\lambda,q}$ is given by

$$R^{\tilde{d}_V^{\lambda,q}} = R^{\tilde{\mathcal{D}}_V} + d^{\tilde{\mathcal{D}}_V}\delta_q^\lambda + \frac{1}{2}[\delta_q^\lambda \wedge \delta_q^\lambda],$$

for $\delta_q^\lambda := \tilde{d}_V^{\lambda,q} - \tilde{\mathcal{D}}_V$. Since there are no non-zero $(2,0)$- or $(0,2)$-forms over a surface,

$$\frac{1}{2}[\delta_q^\lambda \wedge \delta_q^\lambda] = [\tilde{\mathcal{N}}_V^{1,0} \wedge \tilde{\mathcal{N}}_V^{0,1}] + (\lambda^{-1}-\lambda)([q^{1,0} \wedge \tilde{\mathcal{N}}_V^{0,1}] - [q^{0,1} \wedge \tilde{\mathcal{N}}_V^{1,0}]) + (2-\lambda^{-2}-\lambda^2)[q^{1,0} \wedge q^{0,1}],$$

and Gauss-Ricci equation establishes then

$$R^{\tilde{d}_V^{\lambda,q}} = d^{\tilde{\mathcal{D}}_V}\delta_q^\lambda + (\lambda^{-1}-\lambda)([q^{1,0} \wedge \tilde{\mathcal{N}}_V^{0,1}] - [q^{0,1} \wedge \tilde{\mathcal{N}}_V^{1,0}]) + \frac{1}{2}(2-\lambda^{-2}-\lambda^2)[q \wedge q].$$

In its turn, Codazzi equation gives

$$d^{\tilde{\mathcal{D}}_V}\tilde{\mathcal{N}}_V^{1,0}=\frac{i}{2}\,d^{\tilde{\mathcal{D}}_V}*\tilde{\mathcal{N}}_V=-d^{\tilde{\mathcal{D}}_V}\tilde{\mathcal{N}}_V^{0,1},$$

We conclude that

$$\begin{aligned}R^{\tilde{d}_V^{\lambda,q}} &= \frac{\lambda^{-1}-\lambda}{2}\,i\,(d^{\tilde{\mathcal{D}}_V}*\tilde{\mathcal{N}}_V-2[q\wedge *\tilde{\mathcal{N}}_V])\\ &\quad+(\lambda^{-2}-1)\,d^{\tilde{\mathcal{D}}_V}q^{1,0}+(\lambda^2-1)\,d^{\tilde{\mathcal{D}}_V}q^{0,1}+\frac{1}{2}\,(2-\lambda^{-2}-\lambda^2)\,[q\wedge q].\end{aligned}$$

By (6.1) and (6.2), it follows that $R^{\tilde{d}_V^{\lambda,q}}=0$ if and only if both

$$\frac{\lambda^{-1}-\lambda}{2}\,i\,(d^{\tilde{\mathcal{D}}_V}*\tilde{\mathcal{N}}_V-2[q\wedge *\tilde{\mathcal{N}}_V])=0 \tag{6.6}$$

and

$$(\lambda^{-2}-1)\,d^{\tilde{\mathcal{D}}_V}q^{1,0}+(\lambda^2-1)\,d^{\tilde{\mathcal{D}}_V}q^{0,1}+\frac{1}{2}\,(2-\lambda^{-2}-\lambda^2)\,[q\wedge q]=0 \tag{6.7}$$

hold. Organizing equations (6.6) and (6.7) by powers of λ leads us to the final conclusion of the equivalence between the flatness of $\tilde{d}_V^{\lambda,q}$ for all $\lambda\in\mathbb{C}\backslash\{0\}$ and the $(q,\tilde{d})$-constrained complex-harmonicity of V. □

Remark 6.7. *From the proof of Theorem 6.6, we readily verify that the $(q,\tilde{d})$-constrained harmonicity of V is equivalently characterized by the flatness of $\tilde{d}_V^{\lambda,q}$ for all $\lambda\in S^1$.*

Theorem 6.6 will play a crucial role in what follows in the chapter.

6.1.1. Spectral deformation of constrained harmonic bundles. Having observed that the $(q,\tilde{d})$-constrained harmonicity of V ensures the flatness of the metric connection $\tilde{d}_V^{\lambda,q}$ on $\underline{\mathbb{C}}^{n+2}$, it is natural to ask about the $\tilde{d}_V^{\lambda,q}$-constrained harmonicity of V.

Theorem 6.8. *If V is $(q,\tilde{d})$-constrained harmonic then V is $(q_\lambda,\tilde{d}_V^{\lambda,q})$-constrained harmonic, for all $\lambda\in\mathbb{C}\backslash\{0\}$.*

Proof. It is immediate from Theorem 6.6 and equation (6.3). □

Corollary 6.9. *If V is $\tilde{d}$-harmonic then V is $\tilde{d}_V^{\lambda}$-harmonic for any $\lambda\in\mathbb{C}\backslash\{0\}$.*

For a general flat metric connection $\hat{d}$ on $\underline{\mathbb{C}}^{n+2}$, the $\hat{d}$-harmonicity of V follows from its $\tilde{d}$-harmonicity if $V=\phi V$ for some isomorphism $\phi:(\underline{\mathbb{C}}^{n+2},\tilde{d})\to(\underline{\mathbb{C}}^{n+2},\hat{d})$, as established, in particular, in the following theorem, which constitutes a simple, yet crucial, result.

Theorem 6.10. *Let $\hat{d}$ be a flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and*

$$\phi:(\underline{\mathbb{C}}^{n+2},\tilde{d})\to(\underline{\mathbb{C}}^{n+2},\hat{d})$$

be an isomorphism. The bundle V is $(q,\tilde{d})$-constrained harmonic if and only if ϕV is $(\mathrm{Ad}_\phi\, q,\hat{d})$-constrained harmonic.

PROOF. It is immediate from theorem 6.6 and equation (6.5). $\square$

Theorem 6.10 combines with Theorem 6.8 to provide the definition, up to isomorphisms of bundles with a metric and a connection, of a $\mathbb{C}\backslash\{0\}$-deformation of $\tilde{d}$-constrained harmonic bundles. In fact, if V is $\tilde{d}$-constrained harmonic with multiplier q, then so is the transformation $\tilde{\phi}^q_\lambda V$ of V, for $\tilde{\phi}^q_\lambda : (\underline{\mathbb{C}}^{n+2}, \tilde{d}^{\lambda,q}_V) \to (\underline{\mathbb{C}}^{n+2}, \tilde{d})$ an isomorphism and λ in $\mathbb{C}\backslash\{0\}$. Note that this spectral deformation of $\tilde{d}$-constrained harmonic bundles provides, in particular, a $\mathbb{C}\backslash\{0\}$-deformation of $\tilde{d}$-harmonic bundles into $\tilde{d}$-harmonic bundles.

6.2. Complexified surfaces

The transformations of a constrained Willmore surface Λ in the projectivized light-cone we present in this chapter are, in particular, pairs $((\Lambda^{1,0})^*, (\Lambda^{0,1})^*)$ of transformations $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ of $\Lambda^{1,0}$ and $\Lambda^{0,1}$, respectively. The fact that $\Lambda^{1,0}$ and $\Lambda^{0,1}$ intersect in a rank 1 bundle will ensure that $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ have the same property. The isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ will ensure that of $(\Lambda^{1,0})^*$ and $(\Lambda^{0,1})^*$ and, therefore, of their intersection. The reality of the bundle $\Lambda^{1,0}\cap\Lambda^{0,1}$ and the fact that it defines an immersion of M into $\mathbb{P}(\mathcal{L})$ are preserved by the spectral deformation, but it is not clear that the same is necessarily true for Bäcklund transformation. This motivates us to define *complexified surface*.

In the case $\tilde{d}$ is real, and given a $\tilde{d}$-surface Λ, set

$$\Lambda^{1,0}_{\tilde{d}} := \langle \sigma, \tilde{d}_{\delta_z}\sigma\rangle, \qquad \Lambda^{0,1}_{\tilde{d}} := \langle \sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma\rangle = \overline{\Lambda^{1,0}_{\tilde{d}}},$$

independently of the choices of a never-zero section σ of Λ and of a holomorphic chart z of $(M, \mathcal{C}^{\tilde{d}}_\Lambda)$, defining in this way two subbundles of the bundle

$$\Lambda^{(1)}_{\tilde{d}} = \langle \sigma, \tilde{d}_{\delta_z}\sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma\rangle$$

in

$$S^{\tilde{d}} = \langle \sigma, \tilde{d}_{\delta_z}\sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma, \tilde{d}_{\delta_{\bar{z}}}\tilde{d}_{\delta_z}\sigma\rangle \subset \underline{\mathbb{C}}^{n+2},$$

the complexification of the $\tilde{d}$-central sphere congruence of Λ. The $\tilde{d}$-surface condition on Λ, $\mathrm{rank}_{\mathbb{C}}\,\Lambda^{(1)}_{\tilde{d}} = 3$, shows that $\Lambda^{1,0}_{\tilde{d}}$ and $\Lambda^{0,1}_{\tilde{d}}$ are complex rank 2 bundles. On the other hand, the fact that $\tilde{d}$ is metric connection gives

$$(\tilde{d}_{\delta_z}\sigma, \sigma) = 0 = (\tilde{d}_{\delta_{\bar{z}}}\sigma, \sigma),$$

whereas the conformality of $\tilde{\phi}\sigma : (M, \mathcal{C}_{\tilde{\phi}\sigma} = \mathcal{C}^{\tilde{d}}_{\Lambda}) \to \mathbb{R}^{n+1,1}$, fixing an isomorphism $\tilde{\phi} : (\mathbb{R}^{n+1,1}, \tilde{d}) \to (\mathbb{R}^{n+1,1}, d)$, gives

$$(\tilde{d}_{\delta_z}\sigma, \tilde{d}_{\delta_z}\sigma) = 0 = (\tilde{d}_{\delta_{\bar{z}}}\sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma).$$

We conclude that $\Lambda^{1,0}_{\tilde{d}}$ and $\Lambda^{0,1}_{\tilde{d}}$ are isotropic. The fact that $S^{\tilde{d}}$ has complex rank 4 shows that $\Lambda^{1,0}_{\tilde{d}}$ and $\Lambda^{0,1}_{\tilde{d}}$ intersect as the complexification

$$\Lambda = \Lambda^{1,0}_{\tilde{d}} \cap \Lambda^{0,1}_{\tilde{d}},$$

of Λ.

Notation: given $i \neq j \in \{0, 1\}$, $\tilde{d}^{i,j} := \tilde{d}|_{\Gamma(T^{i,j}M)}$.

Definition 6.11. *We define a* complexified $\tilde{d}$-surface *to be a pair* $(\Delta^{1,0}, \Delta^{0,1})$ *of isotropic rank* 2 *subbundles of* $\underline{\mathbb{C}}^{n+2}$ *intersecting in a rank* 1 *bundle*

$$\Delta := \Delta^{1,0} \cap \Delta^{0,1}$$

such that

$$\tilde{d}^{1,0}\Gamma(\Delta) \subset \Omega^{1,0}(\Delta^{1,0}), \quad \tilde{d}^{0,1}\Gamma(\Delta) \subset \Omega^{0,1}(\Delta^{0,1}). \tag{6.8}$$

In the particular case of $\tilde{d} = d$, we shall, alternatively, omit the reference to $\tilde{d}$.

It is, perhaps, worth remarking that a complexified surface does not necessarily define an immersion Δ of M in $\mathbb{P}(\mathcal{L})$.

Obviously, if $\tilde{d}$ is real, then, given a $\tilde{d}$-surface Λ, the pair $(\Lambda^{1,0}_{\tilde{d}}, \Lambda^{0,1}_{\tilde{d}})$ consists of a complexified $\tilde{d}$-surface with respect to $\mathcal{C}^{\tilde{d}}_{\Lambda}$. Observe that the isotropy of $\Delta^{1,0}$ establishes, in particular,

$$(\tilde{d}^{1,0}\sigma, \tilde{d}^{1,0}\sigma) = 0, \tag{6.9}$$

fixing $\sigma \in \Gamma(\Delta)$ never-zero. In the particular case $\tilde{d}$ is real and Δ is a $\tilde{d}$-surface, (we may refer to $\mathcal{C}^{\tilde{d}}_{\Delta}$ and) equation (6.9) is equivalent to

$$\mathcal{C} = \mathcal{C}^{\tilde{d}}_{\Delta},$$

which, together with $\Delta^{1,0} \supset \langle\sigma\rangle + \tilde{d}^{1,0}\sigma(TM)$ and with $\Delta^{0,1} \supset \langle\sigma\rangle + \tilde{d}^{0,1}\sigma(TM)$, shows that

$$\Delta^{1,0} = \Delta^{1,0}_{\tilde{d}}, \quad \Delta^{0,1} = \Delta^{0,1}_{\tilde{d}}$$

and, in particular, that $(\Delta^{1,0}, \Delta^{0,1})$ is uniquely determined by $\Delta^{1,0} \cap \Delta^{0,1}$. We conclude that, for $\tilde{d}$ real and under the correspondence given by

$$(\Delta^{1,0}, \Delta^{0,1}) \leftrightarrow \Delta^{1,0} \cap \Delta^{0,1},$$

$\tilde{d}$-surfaces are the complexified $\tilde{d}$-surfaces given by a pair intersecting as a real line subbundle Δ of $\underline{\mathbb{C}}^{n+2}$,

$$\Delta = \langle\sigma\rangle^{\mathbb{C}},$$

for some $\sigma \in \Gamma(\underline{\mathbb{R}}^{n+1,1})$, such that

$$\mathrm{rank}(\langle\sigma\rangle + \tilde{d}^{1,0}\sigma(TM) + \tilde{d}^{0,1}\sigma(TM)) = 3.$$

In particular, d-surfaces are the complexified d-surfaces given by a pair intersecting as a real line subbundle of $\underline{\mathbb{C}}^{n+2}$ defining an immersion of M into the projectivized light-cone. Henceforth, we drop the term "complexified", referring, when necessary, to *real* $\tilde{d}$-surfaces, in order to make a distinction.

In what follows in this section, let $(\Delta^{1,0}, \Delta^{0,1})$ be a $\tilde{d}$-surface.

Definition 6.12. *A non-degenerate rank* 4 *subbundle* V *of* $\underline{\mathbb{C}}^{n+2}$ *is said to be an* enveloping sphere congruence *of* $(\Delta^{1,0}, \Delta^{0,1})$ *if* $\Delta^{1,0} + \Delta^{0,1} \subset V$.

Note that, if V is an enveloping sphere congruence of $(\Delta^{1,0}, \Delta^{0,1})$, then

$$\mathcal{N}_V^{\tilde{d}}\,\Delta = 0. \tag{6.10}$$

Proposition 6.13. *If* V *is an enveloping sphere congruence of* $(\Delta^{1,0}, \Delta^{0,1})$*, then*

$$(\mathcal{D}_V^{\tilde{d}})^{1,0}\Gamma(\Delta^{1,0}) \subset \Omega^{1,0}(\Delta^{1,0}), \quad (\mathcal{D}_V^{\tilde{d}})^{0,1}\,\Gamma(\Delta^{0,1}) \subset \Omega^{0,1}(\Delta^{0,1}).$$

Before proceeding to the proof of the proposition, observe that, if V is an enveloping sphere congruence of $(\Delta^{1,0}, \Delta^{0,1})$, then $\mathrm{rank}\,\Delta^{1,0} = \frac{1}{2}\,\mathrm{rank}\,V = \mathrm{rank}\,\Delta^{0,1}$, which, together with the isotropy of $\Delta^{1,0}$ and $\Delta^{0,1}$, and having in consideration the non-degeneracy of V, gives

$$(\Delta^{1,0})^{\perp} \cap V = \Delta^{1,0}, \;\; (\Delta^{0,1})^{\perp} \cap V = \Delta^{0,1},$$

$\Delta^{1,0}$ and $\Delta^{0,1}$ are maximal isotropic in V.

Now we proceed to the proof of Proposition 6.13.

Proof. Condition (6.8), together with (6.10), establishes $(\mathcal{D}_V^{\tilde{d}})^{1,0}\,\Gamma(\Delta) \subset \Omega^{1,0}(\Delta^{1,0})$ and $(\mathcal{D}_V^{\tilde{d}})^{0,1}\,\Gamma(\Delta) \subset \Omega^{0,1}(\Delta^{0,1})$. Write $\Delta^{1,0} = \langle\sigma, \tau\rangle$ with $\sigma \in \Gamma(\Delta)$ never-zero. To conclude that $(\mathcal{D}_V^{\tilde{d}})^{1,0}\Gamma(\Delta^{1,0}) \subset \Omega^{1,0}(\Delta^{1,0})$, we are left to verify that $(\mathcal{D}_V^{\tilde{d}})^{1,0}\tau$ takes values in $\Delta^{1,0}$, or, equivalently, in $(\Delta^{1,0})^{\perp}$. The fact that $\mathcal{D}_V^{\tilde{d}}$ is metric and $\Delta^{1,0}$ is isotropic establishes

$$((\mathcal{D}_V^{\tilde{d}})^{1,0}\tau, \tau) = \frac{1}{2}\,d^{1,0}(\tau, \tau) = 0$$

and

$$0 = d^{1,0}(\tau, \sigma) = ((\mathcal{D}_V^{\tilde{d}})^{1,0}\tau, \sigma) + (\tau, (\mathcal{D}_V^{\tilde{d}})^{1,0}\sigma),$$

and, therefore, $((\mathcal{D}_V^{\tilde{d}})^{1,0}\tau, \sigma) = 0$. A similar argument applies to $\Delta^{0,1}$, completing the proof. $\square$

Consider, for a moment, a real surface Λ. Let σ be a never-zero section of Λ and z be a holomorphic chart of $(M, \mathcal{C}_\Lambda)$. The central sphere congruence S of Λ is characterized by having rank 4 and satisfying the conditions $\Lambda^{1,0}+\Lambda^{0,1} \subset S$ and $\sigma_{z\bar{z}} \in \Gamma(S)$. In view of

(2.27), the condition $\sigma_{z\bar{z}} \in \Gamma(S)$ can be reformulated as $\mathcal{N}^{1,0}\Lambda^{0,1} = 0$, or, equivalently, $\mathcal{N}^{0,1}\Lambda^{0,1} = 0$. We conclude that the central sphere congruence of Λ is characterized, equivalently, by enveloping Λ, together with satisfying (5.9). This motivates the next definition.

Definition 6.14. *An enveloping sphere congruence V of $(\Delta^{1,0}, \Delta^{0,1})$ is said to be a* $\tilde{d}$-central sphere congruence *of $(\Delta^{1,0}, \Delta^{0,1})$ if*

$$(\mathcal{N}_V^{\tilde{d}})^{1,0}\Delta^{0,1} = 0 = (\mathcal{N}_V^{\tilde{d}})^{0,1}\Delta^{1,0}.$$

As usual, in the particular case of $\tilde{d} = d$, we shall, alternatively, refer to V as a central sphere congruence.

Observe that, if, given $\sigma \in \Gamma(\Delta)$, $(\tilde{d}^{1,0}\sigma, \tilde{d}^{0,1}\sigma)$ is locally never-zero, or, equivalently,

$$(\tilde{d}^{1,0}\sigma, \tilde{d}^{0,1}\sigma) \neq 0, \tag{6.11}$$

at some point, then $(\Delta^{1,0}, \Delta^{0,1})$ admits a unique $\tilde{d}$-central sphere congruence, that is, $V = \Delta^{1,0} + \Delta^{0,1} + \tilde{d}^{1,0}\tilde{d}^{0,1}\sigma(TM \times TM)$. In fact, the centrality of V, equivalent to

$$\tilde{d}^{1,0}\Gamma(\Delta^{0,1}),\, \tilde{d}^{0,1}\Gamma(\Delta^{1,0}) \subset \Omega^1(V),$$

ensures, in particular, that $\tilde{d}^{1,0}\tilde{d}^{0,1}\sigma \in \Omega^2(V)$. In its turn, equation (6.11) ensures that $\tilde{d}^{1,0}\sigma$ does not take values in $\langle\sigma\rangle$, nor does $\tilde{d}^{0,1}\sigma$ in $\langle\sigma, \tilde{d}^{1,0}\sigma\rangle$, and, therefore, $\mathrm{rank}\,(\Delta^{1,0} + \Delta^{0,1}) \geq 3$, leaving us to verify that $\tilde{d}^{1,0}\tilde{d}^{0,1}\sigma$ does not take values in $\Delta^{1,0} + \Delta^{0,1}$. Suppose it did. In that case, $(\tilde{d}^{1,0}\tilde{d}^{0,1}\sigma, \sigma) = 0$, which contradicts

$$(\tilde{d}^{1,0}\tilde{d}^{0,1}\sigma, \sigma) = d^{1,0}(\tilde{d}^{0,1}\sigma, \sigma) - (\tilde{d}^{0,1}\sigma, \tilde{d}^{1,0}\sigma) = -(\tilde{d}^{0,1}\sigma, \tilde{d}^{1,0}\sigma) \neq 0.$$

Observe, on the other hand, that condition (6.11) is equivalent to

$$(\tilde{d}_{\delta_x}\sigma, \tilde{d}_{\delta_x}\sigma) + (\tilde{d}_{\delta_y}\sigma, \tilde{d}_{\delta_y}\sigma) \neq 0,$$

fixing $z = x + iy$ a holomorphic chart of $(M, \mathcal{C})$, and is, therefore, trivially satisfied in the particular case $\tilde{d}$ is real and Δ is a real $\tilde{d}$-surface, as, in that case, $g_z \in \mathcal{C} = [g_\sigma^{\tilde{d}}]$. It follows, and it is worth emphasizing, that:

Remark 6.15. *In the particular case $\tilde{d}$ is real and $(\Delta^{1,0}, \Delta^{0,1})$ defines a real $\tilde{d}$-surface $\Delta^{1,0} \cap \Delta^{0,1} =: \Delta$, not only $\mathcal{C}$, $\Delta^{1,0}$ and $\Delta^{0,1}$ are determined by Δ, as observed previously, but so is V,* the $\tilde{d}$-central sphere congruence *of $(\Delta^{1,0}, \Delta^{0,1})$: it is the complexification of the $\tilde{d}$-central sphere congruence of Δ, $V = S_\Delta^{\tilde{d}}$. In fact, the identities*

$$\Delta^{1,0} = \langle\sigma, \tilde{d}_{\delta_z}\sigma\rangle, \quad \Delta^{0,1} = \langle\sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma\rangle$$

and

$$V = \langle\sigma, \tilde{d}_{\delta_z}\sigma, \tilde{d}_{\delta_{\bar{z}}}\sigma, \tilde{d}_{\delta_{\bar{z}}}\tilde{d}_{\delta_z}\sigma\rangle,$$

for $\sigma \in \Gamma(\Delta)$ never-zero and z a holomorphic chart of $(M,\mathcal{C})$, still hold, at least, locally, in the case of a general $\tilde{d}$-surface $(\Delta^{1,0},\Delta^{0,1})$, provided that, clearly independently of the choice of σ, we have, at some point, $\sigma \wedge \tilde{d}^{1,0}\sigma \neq 0$ and $(\tilde{d}^{1,0}\sigma, \tilde{d}^{0,1}\sigma) \neq 0$.

6.3. Complexified constrained Willmore surfaces

In generalization of the characterization of Willmore surfaces in space-forms in terms of the harmonicity of the central sphere congruence, a surface Λ in the projectivized light-cone is a q-constrained Willmore surface, for some real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, if and only if S is q-constrained harmonic with respect to the conformal structure $\mathcal{C}_\Lambda$. Generalizing the class of constrained Willmore surfaces in space-forms, we define *complexified constrained Willmore surface* by the property of admitting a constrained harmonic central sphere congruence with a multiplier satisfying certain specificities, as presented in this section.

6.3.1. Complexified constrained Willmore surfaces and constrained harmonicity. Note that, given a real surface Λ, and in view of the fact that $\operatorname{rank}\Lambda = 1$, we have, according to (2.17),

$$[\Lambda \wedge \Lambda^{(1)}, \Lambda \wedge \Lambda^{(1)}] \subset \Lambda \wedge \Lambda = \{0\}.$$

In particular, if q is a multiplier to Λ, then $[q \wedge q] = 0$. Furthermore, according to Lemma 5.10, $d^{\mathcal{D}}q = 0$ if and only if $d^{\mathcal{D}}q^{1,0} = 0 = d^{\mathcal{D}}q^{0,1}$, considering $(1,0)$- and $(0,1)$-decomposition with respect to $\mathcal{C}_\Lambda$. In generalization of Theorem 4.10, it follows that:

Theorem 6.16. *A real surface Λ is q-constrained Willmore, for some real form $q \in \Omega^1(\Lambda\wedge\Lambda^{(1)})$, if and only if its central sphere congruence is q-constrained harmonic with respect to the conformal structure $\mathcal{C}_\Lambda$.*

By Theorem 6.6, together with Remark 6.7, it follows, in generalization of Theorem 4.20, that:

Theorem 6.17. *A real surface Λ is q-constrained Willmore, for some real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, if and only if, considering $(1,0)$- and $(0,1)$-decomposition with respect to $\mathcal{C}_\Lambda$, d^λ_q is a flat connection, for each $\lambda \in \mathbb{C}\backslash\{0\}$ or, equivalently, for each $\lambda \in S^1$.*

As referred previously, Theorem 6.17 was originally established by F. Burstall and D. Calderbank [**11**].

Remark 6.18. *Theorem 5.6 and Lemma 5.10 combine to establish, in particular, that if q is a multiplier to a real surface Λ, then, considering $(1,0)$- and $(0,1)$-decomposition with respect to $\mathcal{C}_\Lambda$, $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$, or, equivalently, $q^{0,1} \in \Omega^{0,1}(\Lambda \wedge \Lambda^{1,0})$.*

In view of Theorem 6.16, we generalize the class of constrained Willmore surfaces in space-forms with the following definition:

Definition 6.19. *A $\tilde{d}$-surface $(\Delta^{1,0}, \Delta^{0,1})$ is said to be a constrained Willmore $\tilde{d}$-surface if there exist*

$$q^{1,0} \in \Omega^{1,0}(\wedge^2 \Delta^{0,1}), \quad q^{0,1} \in \Omega^{0,1}(\wedge^2 \Delta^{1,0}) \tag{6.12}$$

for which, setting $q := q^{1,0} + q^{0,1}$, $(\Delta^{1,0}, \Delta^{0,1})$ admits a $(q, \tilde{d})$-constrained harmonic $\tilde{d}$-central sphere congruence.

In the conditions of the definition above, we may refer to $(\Delta^{1,0}, \Delta^{0,1})$ as a $(q, \tilde{d})$-*constrained Willmore surface*. In the particular case $\tilde{d}$ is real, $(\Delta^{1,0}, \Delta^{0,1})$ is a real $\tilde{d}$-surface and q is real, we may refer to $(\Delta^{1,0}, \Delta^{0,1})$ as a real constrained Willmore $\tilde{d}$-surface or a real $(q, \tilde{d})$-constrained Willmore surface. For simplicity, we may, alternatively, refer to a $(q, \tilde{d})$-constrained harmonic $\tilde{d}$-central sphere congruence as a $(q, \tilde{d})$-*central sphere congruence*. The form q is said to be a *multiplier* to the constrained Willmore $\tilde{d}$-surface $(\Delta^{1,0}, \Delta^{0,1})$. In the particular case $\tilde{d} = d$, we shall omit the reference to $\tilde{d}$ and refer to $(\Delta^{1,0}, \Delta^{0,1})$ as a constrained Willmore surface or a q-constrained Willmore surface, or, in the case $\tilde{d}$, $(\Delta^{1,0}, \Delta^{0,1})$ and q are real, as a real q-constrained Willmore surface. In the light of this terminology, the surfaces studied in Chapter 5 shall be renamed *real constrained Willmore surfaces*. In the particular case $q = 0$ we may refer to $(\Delta^{1,0}, \Delta^{0,1})$ as a Willmore $\tilde{d}$-surface, in line with Definition 4.18; or, alternatively, in the case $\tilde{d}$ is real and $(\Delta^{1,0}, \Delta^{0,1})$ is a real $\tilde{d}$-surface, as a *real Willmore $\tilde{d}$-surface*. When referring to a surface in the projectivized light-cone, or, equivalently, in some space-form, it shall be understood a real surface, with no need to express it.

6.3.2. Complexified constrained Willmore surfaces under change of flat metric connection. Let $(\Delta^{1,0}, \Delta^{0,1})$ be a pair of isotropic rank 2 subbundles of $\underline{\mathbb{C}}^{n+2}$, intersecting in a rank 1 bundle, and

$$\tilde{\phi} : (\underline{\mathbb{C}}^{n+2}, \tilde{d}) \to (\underline{\mathbb{C}}^{n+2}, d)$$

be an isomorphism. Obviously, $(\tilde{\phi}\,\Delta^{1,0}, \tilde{\phi}\,\Delta^{0,1})$ is another pair of isotropic rank 2 subbundles of $\underline{\mathbb{C}}^{n+2}$. It is clear that $(\Delta^{1,0}, \Delta^{0,1})$ is a $\tilde{d}$-surface if and only if $(\tilde{\phi}\,\Delta^{1,0}, \tilde{\phi}\,\Delta^{0,1})$ is a d-surface, and that, according to (6.4), given a $\tilde{d}$-central sphere congruence V of $(\Delta^{1,0}, \Delta^{0,1})$, $\tilde{\phi}\,V$ is a central sphere congruence of $(\tilde{\phi}\,\Delta^{1,0}, \tilde{\phi}\,\Delta^{0,1})$. Furthermore:

Proposition 6.20. *Suppose $(\Delta^{1,0}, \Delta^{0,1})$ is a $\tilde{d}$-surface. In that case, $(\Delta^{1,0}, \Delta^{0,1})$ is a $(q, \tilde{d})$-constrained Willmore surface admitting V as a $(q, \tilde{d})$-central sphere congruence if and only if $(\tilde{\phi}\,\Delta^{1,0}, \tilde{\phi}\,\Delta^{0,1})$ is a $\mathrm{Ad}_{\tilde{\phi}}\, q$-constrained Willmore surface admitting $\tilde{\phi}\,V$ as a $\mathrm{Ad}_{\tilde{\phi}}\, q$-central sphere congruence.*

PROOF. According to equation (2.18), $\mathrm{Ad}_{\tilde{\phi}}\, q^{i,j} \in \Omega^{i,j}(\wedge^2 \tilde{\phi}\, \Delta^{j,i})$, for $i \neq j \in \{0,1\}$. The result comes as an immediate consequence of Theorem 6.10. $\square$

If $\tilde{d}$ is real, one can take $\tilde{\phi}$ to be real, in which case $(\Delta^{1,0}, \Delta^{0,1})$ is a real $\tilde{d}$-surface if and only if $(\tilde{\phi}\, \Delta^{1,0}, \tilde{\phi}\, \Delta^{0,1})$ is a real d-surface:

$$\overline{\tilde{\phi}\, \Delta^{1,0} \cap \tilde{\phi}\, \Delta^{0,1}} = \tilde{\phi}\, \overline{\Delta^{1,0} \cap \Delta^{0,1}}$$

and, given $\sigma \in \Gamma(\Delta^{1,0} \cap \Delta^{0,1})$ never-zero,

$$\langle \tilde{\phi}\sigma \rangle + d^{1,0}(\tilde{\phi}\sigma)(TM) + d^{0,1}(\tilde{\phi}\sigma)(TM) = \tilde{\phi}(\langle \sigma \rangle + \tilde{d}^{1,0}\sigma(TM) + \tilde{d}^{0,1}\sigma(TM)).$$

Furthermore, $(\Delta^{1,0}, \Delta^{0,1})$ is a real $(q, \tilde{d})$-constrained Willmore surface if and only if $(\tilde{\phi}\, \Delta^{1,0}, \tilde{\phi}\, \Delta^{0,1})$ is a real $\mathrm{Ad}_{\tilde{\phi}}\, q$-constrained Willmore surface.

6.4. Spectral deformation of complexified constrained Willmore surfaces

Complexified constrained Willmore surfaces are characterized by the flatness of the metric connection $d_V^{\lambda,q}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$, for some central sphere congruence V and some $q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$ satisfying certain specificities. Given a complexified constrained Willmore surface, such family of flat metric connections provides a spectral deformation of the surface into new complexified constrained Willmore surfaces, which, in the case of a complexified Willmore surface and of the zero multiplier, remains within the class of complexified Willmore surfaces, and, for $\lambda \in S^1$, preserves reality conditions. The deformation defined by the loop of flat metric connections d_q^λ coincides, up to reparametrization, with the spectral deformation of a q-constrained Willmore surface in spherical space presented in [**14**].

Suppose $(\Delta^{1,0}, \Delta^{0,1})$ is a constrained Willmore surface admitting q as a multiplier and V as a (q, d)-central sphere congruence, in which case, according to Theorem 6.6, the $\mathbb{C}\backslash\{0\}$-family

$$d_V^{\lambda,q} = \mathcal{D}_V + \lambda^{-1}\mathcal{N}_V^{1,0} + \lambda\mathcal{N}_V^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1},$$

on $\lambda \in \mathbb{C}\backslash\{0\}$, consists of a family of flat metric connections on $\underline{\mathbb{C}}^{n+2}$. Consider an isomorphism

$$\phi_q^\lambda : (\underline{\mathbb{C}}^{n+2}, d_V^{\lambda,q}) \to (\underline{\mathbb{C}}^{n+2}, d).$$

Since $q^{i,j}$ takes values in $\wedge^2 \Delta^{j,i}$ and $\Delta^{j,i}$ is isotropic, for $i \neq j \in \{0,1\}$, we have $q^{1,0}\Delta = 0 = q^{0,1}\Delta$, which, together with (6.10), shows that, given $\sigma \in \Gamma(\Delta)$ never-zero, $d_V^{\lambda,q}\sigma = \mathcal{D}_V\sigma$. Equation (6.10) establishes, on the other hand, $\mathcal{D}_V\sigma = d\sigma$. Hence

$$d^{i,j}(\phi_q^\lambda \sigma) = \phi_q^\lambda((d_V^{\lambda,q})^{i,j}\sigma) = \phi_q^\lambda(d^{i,j}\sigma),$$

for $i \neq j \in \{0,1\}$, leading us to conclude that $(\phi_q^\lambda\, \Delta^{1,0}, \phi_q^\lambda\, \Delta^{0,1})$ is still a d-surface. Furthermore:

Theorem 6.21. *If $(\Delta^{1,0},\Delta^{0,1})$ is a q-constrained Willmore surface admitting V as a (q,d)-central sphere congruence, then $(\phi_q^\lambda\,\Delta^{1,0},\phi_q^\lambda\,\Delta^{0,1})$ is a $\mathrm{Ad}_{\phi_q^\lambda}(q_\lambda)$-constrained Willmore surface admitting $\phi_q^\lambda\,V$ as a $(\mathrm{Ad}_{\phi_q^\lambda}(q_\lambda),d)$-central sphere congruence, for each $\lambda\in\mathbb{C}\backslash\{0\}$.*

The proof will consist of showing that, if $(\Delta^{1,0},\Delta^{0,1})$ is a q-constrained Willmore surface admitting V as a (q,d)-central sphere congruence, then $(\Delta^{1,0},\Delta^{0,1})$ is also a $(q_\lambda,d_V^{\lambda,q})$-constrained Willmore surface admitting V as a $(q_\lambda,d_V^{\lambda,q})$-central sphere congruence, for each $\lambda\in\mathbb{C}\backslash\{0\}$.

Proof. Fix $\lambda\in\mathbb{C}\backslash\{0\}$. The crucial fact that

$$d_V^{\lambda,q}\sigma=d\sigma,\tag{6.13}$$

for $\sigma\in\Gamma(\Delta)$ never-zero, shows that, if $(\Delta^{1,0},\Delta^{0,1})$ is a surface, then $(\Delta^{1,0},\Delta^{0,1})$ is also a $d_V^{\lambda,q}$-surface. On the other hand, as $q^{1,0},q^{0,1}\in\Omega^1(\wedge^2V)$ and V and $V^\perp$ are $\mathcal{D}$-parallel, we have

$$\mathcal{N}_V^{d_V^{\lambda,q}}=\lambda^{-1}\mathcal{N}_V^{1,0}+\lambda\mathcal{N}_V^{0,1},$$

making clear that, if V is a central sphere congruence of $(\Delta^{1,0},\Delta^{0,1})$, then V is, as well, a $d_V^{\lambda,q}$-central sphere congruence of $(\Delta^{1,0},\Delta^{0,1})$. Furthermore: according to Theorem 6.8, the (q,d)-constrained harmonicity of V ensures its $(q_\lambda,d_V^{\lambda,q})$-constrained harmonicity. Since $q_\lambda^{1,0}$ and $q_\lambda^{0,1}$ are scales of, respectively, $q^{1,0}$ and $q^{0,1}$, we conclude that $q_\lambda^{1,0}\in\Omega^{1,0}(\wedge^2\Delta^{0,1})$ and $q_\lambda^{0,1}\in\Omega^{0,1}(\wedge^2\Delta^{1,0})$. The result follows now from Proposition 6.20. $\square$

The q-constrained harmonicity of V, characterized by the flatness of $d_V^{\lambda,q}$, for all $\lambda\in\mathbb{C}\backslash\{0\}$, establishes, equivalently, the flatness of

$$d_{\phi_q^\lambda V}^{\mu,\mathrm{Ad}_{\phi_q^\lambda}q_\lambda}=\phi_q^\lambda\circ(d_V^{\lambda,q})_V^{\mu,q_\lambda}\circ(\phi_q^\lambda)^{-1}=\phi_q^\lambda\circ d_V^{\lambda\mu,q}\circ(\phi_q^\lambda)^{-1},$$

for all $\lambda,\mu\in\mathbb{C}\backslash\{0\}$, or, equivalently, the $\mathrm{Ad}_{\phi_q^\lambda}q_\lambda$-harmonicity of $\phi_q^\lambda V$, for all λ. On the other hand, for each $\lambda\in\mathbb{C}\backslash\{0\}$, the deformation $\phi_q^\lambda V$ of V is a central sphere congruence to the deformation $(\phi_q^\lambda\,\Delta^{1,0},\phi_q^\lambda\,\Delta^{0,1})$ of $(\Delta^{1,0},\Delta^{0,1})$. For each $\lambda\in\mathbb{C}\backslash\{0\}$, the flat metric connection $d_V^{\lambda,q}$ provides, in this way, a deformation of the constrained Willmore surface $(\Delta^{1,0},\Delta^{0,1})$ into another constrained Willmore surface. Note that in the case $(\Delta^{1,0},\Delta^{0,1})$ is a Willmore surface and $q=0$, $(\phi_q^\lambda\,\Delta^{1,0},\phi_q^\lambda\,\Delta^{0,1})$ is still a Willmore surface. Such a $\mathbb{C}\backslash\{0\}$-spectral deformation of constrained Willmore surfaces provides, in particular, a S^1-spectral deformation of real constrained Willmore surfaces, as we observe next.

6.4.1. Real spectral deformation. Suppose that $(\Delta^{1,0},\Delta^{0,1})$ defines a real q-constrained Willmore surface $\Lambda:=\Delta^{1,0}\cap\Delta^{0,1}$. In that case, cf. Remark 6.5, for each $\lambda\in S^1$, $d_V^{\lambda,q}$ defines a connection on $\underline{\mathbb{R}}^{n+1,1}$, so that we can take ϕ_q^λ to be real. Fix

$\lambda \in S^1$. Consider ϕ_q^λ to be real, i.e., ϕ_q^λ the complex linear extension to $\underline{\mathbb{C}}^{n+2}$ of an isomorphism

$$\phi_q^\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_V^{\lambda,q}) \to (\underline{\mathbb{R}}^{n+1,1}, d).$$

Then, obviously, $\overline{\phi_q^\lambda \Lambda} = \phi_q^\lambda \Lambda$. On the other hand, given $\sigma \in \Gamma(\Lambda)$ never-zero, the crucial equation (6.13) gives

$$\langle \phi_q^\lambda \sigma \rangle + d^{1,0}(\phi_q^\lambda \sigma)(TM) + d^{0,1}(\phi_q^\lambda \sigma)(TM) = \langle \sigma \rangle + d^{1,0}\sigma(TM) + d^{1,0}\sigma(TM).$$

We conclude that $(\phi_q^\lambda \Delta^{1,0}, \phi_q^\lambda \Delta^{0,1})$ still defines a real surface,

$$\phi_q^\lambda \Lambda =: \Lambda_q^\lambda,$$

which we denote by the *spectral deformation of parameter* λ *of* Λ*, corresponding to the multiplier* q.

For $\lambda \in S^1$, the deformation of $(\Delta^{1,0}, \Delta^{0,1})$ provided by $d_V^{\lambda,q}$ preserves reality conditions. It preserves, as well, the conformal structure induced in M: yet again according to equation (6.13), given $\sigma \in \Gamma(\Lambda)$ never-zero, $g_\sigma^{d_V^{\lambda,q}} = g_\sigma$ and, therefore,

$$\mathcal{C}_{\Lambda_q^\lambda} = \mathcal{C}_\Lambda.$$

According to Theorem 6.21, it preserves the central sphere congruence, as well,

$$S_{\phi_q^\lambda \Lambda} = \phi_q^\lambda S_\Lambda. \tag{6.14}$$

Following Theorem 6.21, we have:

Theorem 6.22. *Suppose that* Λ *is a real* q*-constrained Willmore surface, for some* $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$*. Then, for each* $\lambda \in S^1$*, the transformation* $\phi_q^\lambda \Lambda$ *of* Λ *defined by the flat metric connection* d_q^λ *is a real* $\mathrm{Ad}_{\phi_q^\lambda}(q_\lambda)$*-constrained Willmore surface.*

The loop of flat metric connections d_q^λ defines in this way a S^1-deformation of the real q-constrained Willmore surface Λ into a family of real constrained Willmore surfaces. In the particular case Λ is Willmore and $q = 0$, the deformation remains within the class of Willmore surfaces; in fact, it coincides with the deformation presented in Section 4.7.

An alternative perspective on this spectral deformation of the real q-constrained Willmore surface Λ is that of the action

$$\Lambda \subset (\underline{\mathbb{R}}^{n+1,1}, d) \mapsto \Lambda \subset (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda),$$

of the loop of flat metric connections d_q^λ on $\{\Lambda\}$, consisting of the change of the trivial flat connection on $\underline{\mathbb{R}}^{n+1,1}$ into the flat metric connection d_q^λ, for each λ. Equation (6.13) establishes $\Lambda^{(1)}_{d_q^\lambda} = \Lambda^{(1)}$ and, therefore, Λ as d_q^λ-surface. Equation (6.3) establishes, furthermore, Λ as a (q_λ, d_q^λ)-constrained Willmore surface.

We complete this section by verifying that the deformation defined by the loop of flat metric connections d^λ_q coincides, up to reparametrization, with the spectral deformation of a q-constrained Willmore surface in spherical space defined in [**14**].[1]Fix a holomorphic chart z of $(M, \mathcal{C}_{\Lambda^\lambda_q}) = (M, \mathcal{C}_\Lambda)$. We have

$$g_{\phi^\lambda_q \sigma^z} = g_{\sigma^z} = g_z,$$

showing that $\phi^\lambda_q \sigma^z$ is the normalized section of $\phi^\lambda_q \Lambda$ with respect to z. Since

$$\begin{aligned}(\phi^\lambda_q \sigma^z)_{zz} &= (\phi^\lambda_q \sigma^z_z)_z \\ &= \phi^\lambda_q((d^\lambda_q)_{\delta_z} \sigma^z_z) \\ &= \phi^\lambda_q(\mathcal{D}_{\delta_z} \sigma^z_z + \lambda^{-1} \mathcal{N}_{\delta_z} \sigma^z_z + (\lambda^{-2} - 1) q_{\delta_z} \sigma^z_z) \\ &= \phi^\lambda_q(\pi_S \sigma^z_{zz} + \lambda^{-1} \pi_{S^\perp} \sigma^z_{zz} - \frac{1}{2}(\lambda^{-2} - 1)\, q^z \sigma^z)\end{aligned}$$

and, ultimately,

$$(\phi^\lambda_q \sigma^z)_{zz} = -\frac{1}{2}\,(c^z + (\lambda^{-2} - 1)\, q^z)\, \phi^\lambda_q \sigma^z + \lambda^{-1} \phi^\lambda_q k^z,$$

we conclude that $(k^\lambda_q)^z$ and $(c^\lambda_q)^z$, the Hopf differential and the Schwarzian derivative, respectively, of $\phi^\lambda_q \Lambda$ with respect to z, relate to those of Λ by

$$(k^\lambda_q)^z = \lambda^{-1} \phi^\lambda_q k^z, \quad (c^\lambda_q)^z = c^z + (\lambda^{-2} - 1)\, q^z.$$

By Lemma 5.5, having in consideration (6.14), the conclusion follows.

6.5. Dressing action

We use a version of the dressing action theory of C.-L. Terng and K. Uhlenbeck [**54**] to build transformations of constrained Willmore surfaces. We start by defining a local action of a group of rational maps on the set of flat metric connections of the type $\hat{d}^{\lambda,q}_S$, with $\hat{d}$ flat metric connection on $\underline{\mathbb{C}}^{n+2}$ and $q \in \Omega^1(\wedge^2 S \oplus \wedge^2 S^\perp)$. Namely, given $r = r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ holomorphic at $\lambda = 0$ and $\lambda = \infty$ and twisted in the sense that $\rho r(\lambda) \rho = r(-\lambda)$, for ρ reflection across S, we define a 1-form $\hat{q}$ with values in $\wedge^2 S$ (note that the fact that $r(\lambda)$ is twisted establishes that both $r(0)$ and $r(\infty)$ preserve S) by $\hat{q}^{1,0} := \mathrm{Ad}_{r(0)} q^{1,0}$, $\hat{q}^{\,0,1} := \mathrm{Ad}_{r(\infty)} q^{0,1}$, and a new family of metric connections from $d^{\lambda,q}_S$ by $\hat{d}^{\lambda,\hat{q}}_S := r(\lambda) \circ d^{\lambda,q}_S \circ r(\lambda)^{-1}$. Obviously, for each λ, the flatness of $\hat{d}^{\lambda,\hat{q}}_S$ is equivalent to that of $d^{\lambda,q}_S$. Crucially, if $\hat{d}^{\lambda,\hat{q}}_S$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$, then the notation $\hat{d}^{\lambda,\hat{q}}_S$ proves to be not merely formal, for $\hat{d} := \hat{d}^{1,\hat{q}}_S$. In that case, it follows that, if Λ is q-constrained Willmore, then S is $(\hat{q}, \hat{d})$-constrained harmonic and, therefore, in the case $1 \in \mathrm{dom}(r)$, $S^* := r(1)^{-1} S$ is q^*-constrained harmonic, for $q^* := \mathrm{Ad}_{r(1)^{-1}} \hat{q}$. The transformation of

[1]The omission, in [**14**], of reference to the transformation rule of the normal connection shall be understood as preservation.

S into S^*, preserving constrained harmonicity, leads, furthermore, to a transformation of Λ into a new constrained Willmore surface, provided that $\det r(0)|_S = \det r(\infty)|_S$. Set $(\Lambda^*)^{1,0} := r(1)^{-1}r(\infty)\Lambda^{1,0}$, $(\Lambda^*)^{0,1} := r(1)^{-1}r(0)\Lambda^{0,1}$ and $\Lambda^* := (\Lambda^*)^{1,0} \cap (\Lambda^*)^{0,1}$. The condition above on the determinants of $r(0)|_S$ and $r(\infty)|_S$ establishes Λ^* as a line bundle (the argument is based on the two families of lines on the Klein quadric). The isotropy of $\Lambda^{1,0}$ and $\Lambda^{0,1}$ ensures that of Λ^*. It is not clear, though, that Λ^* is a real bundle. If Λ^* is a real surface, one proves that S^* is the central sphere congruence of Λ^* and that the bundles $(\Lambda^*)^{1,0}$ and $(\Lambda^*)^{0,1}$ defined above are not merely formal. The fact that $q^{1,0} \in \Omega^{1,0}(\wedge^2\Lambda^{0,1})$ establishes $(q^*)^{1,0} \in \Omega^{1,0}(\wedge^2(\Lambda^*)^{0,1}) \subset \Omega^{1,0}(\Lambda^* \wedge (\Lambda^*)^{(1)})$. We conclude that, if, furthermore, q^* is real, then Λ^* is a q^*-constrained Willmore surface. In fact, we use a version of the dressing action theory of C.-L. Terng and K. Uhlenbeck [**54**] to build, more generally, transformations of constrained harmonic bundles and complexified constrained Willmore surfaces.

Let $\rho \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ be reflection across V,

$$\rho = \pi_V - \pi_{V^\perp}.$$

Obviously, given $w \in \Gamma(\underline{\mathbb{C}}^{n+2})$, w is a section of V (respectively, a section of $V^\perp$) if and only if $\rho w = w$ (respectively, $\rho w = -w$). Note that $\rho^{-1} = \rho$. Let q be a 1-form with values in $\wedge^2 V \oplus \wedge^2 V^\perp$. The $\mathcal{D}_V$-parallelness of V and $V^\perp$, together with the fact that $\mathcal{N}_V$ intertwines V and $V^\perp$, whereas q preserves them, makes clear that

$$d_V^{-\lambda,q} = \rho \circ d_V^{\lambda,q} \circ \rho^{-1}, \tag{6.15}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$. Suppose we have $r(\lambda) \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ such that $\lambda \mapsto r(\lambda)$ is rational in λ, r is holomorphic and invertible at $\lambda = 0$ and $\lambda = \infty$ and twisted in the sense that

$$\rho\, r(\lambda)\, \rho^{-1} = r(-\lambda), \tag{6.16}$$

for $\lambda \in \mathrm{dom}(r)$. In particular, it follows that both $r(0)$ and $r(\infty)$ commute with ρ, and, therefore, that

$$r(0)_{|_V}, r(\infty)_{|_V} \in \Gamma(O(V)), \quad r(0)_{|_{V^\perp}}, r(\infty)_{|_{V^\perp}} \in \Gamma(O(V^\perp)). \tag{6.17}$$

Define $\hat{q} \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$ by setting

$$\hat{q}^{1,0} := \mathrm{Ad}_{r(0)} q^{1,0}, \quad \hat{q}^{\,0,1} := \mathrm{Ad}_{r(\infty)} q^{0,1}.$$

Define a new family of metric connections on $\underline{\mathbb{C}}^{n+2}$ by setting

$$\hat{d}_V^{\lambda,\hat{q}} := r(\lambda) \circ d_V^{\lambda,q} \circ r(\lambda)^{-1}.$$

Suppose that there exists a holomorphic extension of $\lambda \mapsto \hat{d}_V^{\lambda,\hat{q}}$ to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$. In that case, as we, crucially, verify next, the notation $\hat{d}_V^{\lambda,\hat{q}}$ is not merely formal:

Proposition 6.23.

$$\hat{d}_V^{\lambda,\hat{q}} = \mathcal{D}_V^{\hat{d}} + \lambda^{-1}(\mathcal{N}_V^{\hat{d}})^{1,0} + \lambda(\mathcal{N}_V^{\hat{d}})^{0,1} + (\lambda^{-2}-1)\hat{q}^{1,0} + (\lambda^2-1)\hat{q}^{0,1},$$

for the flat metric connection

$$\hat{d} := \hat{d}_V^{1,\hat{q}} = \lim_{\lambda\to 1} r(\lambda)\circ d_V^{\lambda,q}\circ r(\lambda)^{-1}$$

and $\lambda \in \mathbb{C}\backslash\{0\}$.

Proof. First note that, as r is holomorphic and invertible at $\lambda = 0$ and

$$(d_V^{\lambda,q})^{0,1} = \mathcal{D}_V^{0,1} + \lambda\mathcal{N}_V^{0,1} + (\lambda^2-1)q^{0,1}$$

is holomorphic on $\mathbb{C}$, the connection

$$(\hat{d}_V^{\lambda,\hat{q}})^{0,1} = r(\lambda)\circ(d_V^{\lambda,q})^{0,1}\circ r(\lambda)^{-1}$$

which admits a holomorphic extension to $\lambda\in\mathbb{C}\backslash\{0\}$, admits, furthermore, a holomorphic extension to $\lambda\in\mathbb{C}$. Thus, locally,

$$(\hat{d}_V^{\lambda,\hat{q}})^{0,1} = A_0^{0,1} + \sum_{i\geq 1}\lambda^i A_i^{0,1},$$

with A_0 connection and $A_i \in \Omega^1(o(\underline{\mathbb{C}}^{n+2}))$, for all i. Considering then limits of

$$\lambda^{-2}A_0^{0,1} + \sum_{i\geq 1}\lambda^{i-2}A_i^{0,1} = r(\lambda)\circ(\lambda^{-2}\mathcal{D}_V^{0,1} + \lambda^{-1}\mathcal{N}_V^{0,1} + (1-\lambda^{-2})q^{0,1})\circ r(\lambda)^{-1},$$

when λ goes to infinity, we get

$$A_2^{0,1} + \lim_{\lambda\to\infty}\sum_{i\geq 3}\lambda^{i-2}A_i^{0,1} = \mathrm{Ad}_{r(\infty)}\, q^{0,1},$$

which shows that $A_i^{0,1} = 0$, for all $i \geq 3$, and that $A_2^{0,1} = \hat{q}^{0,1}$. Considering now limits of

$$A_0^{0,1} + \lambda A_1^{0,1} + \lambda^2\hat{q}^{0,1} = r(\lambda)\circ(\mathcal{D}_V^{0,1} + \lambda\mathcal{N}_V^{0,1} + (\lambda^2-1)q^{0,1})\circ r(\lambda)^{-1},$$

when λ goes to 0, we conclude that $A_0^{0,1} = r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1})$ and, therefore, that $(\hat{d}_V^{\lambda,\hat{q}})^{0,1} = r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1}) + \lambda A_1^{0,1} + \lambda^2\hat{q}^{0,1}$. As for

$$(\hat{d}_V^{\lambda,\hat{q}})^{1,0} = r(\lambda)\circ(\mathcal{D}_V^{1,0} + \lambda^{-1}\mathcal{N}^{1,0} + (\lambda^{-2}-1)q^{1,0})\circ r(\lambda)^{-1},$$

which has a pole at $\lambda = 0$, we have, for λ away from 0,

$$\sum_{i\geq 1}\lambda^{-i}A_{-i}^{1,0} + A_0^{1,0} + \sum_{i\geq 1}\lambda^i A_i^{1,0} = r(\lambda)\circ(\mathcal{D}_V^{1,0} + \lambda^{-1}\mathcal{N}^{1,0} + (\lambda^{-2}-1)q^{1,0})\circ r(\lambda)^{-1}, \tag{6.18}$$

with $A_{-i}^{1,0} \in \Omega^1(o(\underline{\mathbb{C}}^{n+2}))$, for all $i \geq 1$. Considering limits of (6.18) when λ goes to infinity, shows that $A_i^{1,0} = 0$, for all $i \geq 1$, and that $A_0^{1,0} = r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0})$. Multiplying then both members of equation (6.18) by λ^2 and considering limits when λ goes to 0, we conclude that $A_{-2}^{1,0} = \hat{q}^{1,0}$ and that $A_{-i}^{1,0} = 0$, for all $i \geq 3$, and, ultimately,

that $(\hat{d}_V^{\lambda,\hat{q}})^{1,0} = r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0}) + \lambda^{-1}A_{-1}^{1,0} + \lambda^{-2}\hat{q}^{1,0}$. Thus

$$\begin{aligned}\hat{d}_V^{\lambda,\hat{q}} &= r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1} + q^{1,0}) + r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0} + q^{0,1})\\ &\quad + \lambda^{-1}A_{-1}^{1,0} + \lambda A_1^{0,1} + (\lambda^{-2}-1)\hat{q}^{1,0} + (\lambda^2 - 1)\hat{q}^{0,1},\end{aligned}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$, and, in particular,

$$\hat{d} = r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1} + q^{1,0}) + r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0} + q^{0,1}) + A_{-1}^{1,0} + A_1^{0,1}.$$

The fact that $r(0)$ and $r(\infty)$ (and so $r(0)^{-1}$ and $r(\infty)^{-1}$), as well as q, preserve V and $V^\perp$, together with the $\mathcal{D}_V$-parallelness of V and of $V^\perp$, shows that $\hat{d} - (A_{-1}^{1,0} + A_1^{0,1})$ preserves $\Gamma(V)$ and $\Gamma(V^\perp)$. On the other hand, equations (6.15) and (6.16) combine to give

$$\hat{d}_V^{-\lambda,\hat{q}} = \rho\circ\hat{d}_V^{\lambda,\hat{q}}\circ\rho^{-1},$$

for all $\lambda\in\mathbb{C}\backslash\{0\}$ away from the poles of r and then, by continuity, on all of $\mathbb{C}\backslash\{0\}$. The particular case of $\lambda = 1$ gives

$$\rho(A_{-1}^{1,0} + A_1^{0,1})_{|V} = -(A_{-1}^{1,0} + A_1^{0,1})_{|V},\;\; \rho(A_{-1}^{1,0} + A_1^{0,1})_{|V^\perp} = -(A_{-1}^{1,0} + A_1^{0,1})_{|V^\perp},$$

showing that that $A_{-1}^{1,0} + A_1^{0,1} \in \Omega^1(V\wedge V^\perp)$. We conclude that

$$r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1} + q^{1,0}) + r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0} + q^{0,1}) = \mathcal{D}_V^{\hat{d}} \tag{6.19}$$

and

$$A_{-1}^{1,0} = (\mathcal{N}_V^{\hat{d}})^{1,0}, \quad A_1^{0,1} = (\mathcal{N}_V^{\hat{d}})^{0,1},$$

completing the proof. □

Before proceeding any further, we remark on the $(1,0)$- and $(0,1)$-components of $\mathcal{D}_V^{\hat{d}}$ and $\mathcal{N}_V^{\hat{d}}$. According to (6.19),

$$(\mathcal{D}_V^{\hat{d}})^{1,0} = r(\infty)\cdot(\mathcal{D}_V^{1,0} - q^{1,0}) + \hat{q}^{1,0},\;\; (\mathcal{D}_V^{\hat{d}})^{0,1} = r(0)\cdot(\mathcal{D}_V^{0,1} - q^{0,1}) + \hat{q}^{0,1}. \tag{6.20}$$

As for $\mathcal{N}_V^{\hat{d}}$, according to Proposition 6.23,

$$\begin{aligned}(\mathcal{N}_V^{\hat{d}})^{1,0} &= \lim_{\lambda\to 0}\lambda((\hat{d}_V^{\lambda,\hat{q}})^{1,0} - (\mathcal{D}_V^{\hat{d}})^{1,0} - (\lambda^{-2}-1)\hat{q}^{1,0})\\ &= \lim_{\lambda\to 0}\lambda((\hat{d}_V^{\lambda,\hat{q}})^{1,0} - \lambda^{-2}\hat{q}^{1,0})\\ &= \lim_{\lambda\to 0}(r(\lambda)\circ(\lambda\,(d_V^{\lambda,q})^{1,0})\circ r(\lambda)^{-1} - \lambda^{-1}\mathrm{Ad}_{r(0)}q^{1,0})\\ &= \mathrm{Ad}_{r(0)}\mathcal{N}_V^{1,0} + \lim_{\lambda\to 0}\frac{1}{\lambda}(\mathrm{Ad}_{r(\lambda)} - \mathrm{Ad}_{r(0)})q^{1,0}.\end{aligned}$$

so that

$$(\mathcal{N}_V^{\hat{d}})^{1,0} = \mathrm{Ad}_{r(0)}\mathcal{N}_V^{1,0} + \frac{d}{d\lambda}_{|\lambda=0}\mathrm{Ad}_{r(\lambda)}q^{1,0}; \tag{6.21}$$

and, similarly,

$$
\begin{aligned}
(\mathcal{N}_V^{\hat{d}})^{0,1} &= \lim_{\lambda\to\infty}\lambda^{-1}((\hat{d}_V^{\lambda,\hat{q}})^{0,1} - (\mathcal{D}_V^{\hat{d}})^{0,1} - (\lambda^2-1)\hat{q}^{0,1}) \\
&= \mathrm{Ad}_{r(\infty)}\mathcal{N}_V^{0,1} + \lim_{\lambda\to\infty}(r(\lambda)\circ\lambda q^{0,1}\circ r(\lambda)^{-1} - \lambda\mathrm{Ad}_{r(\infty)}q^{0,1})
\end{aligned}
$$

and, therefore,

$$
(\mathcal{N}_V^{\hat{d}})^{0,1} = \mathrm{Ad}_{r(\infty)}\mathcal{N}_V^{0,1} + \frac{d}{d\lambda}_{|\lambda=0}\mathrm{Ad}_{r(\lambda^{-1})}q^{0,1}. \tag{6.22}
$$

Now suppose V is a q-constrained harmonic bundle, in which case, according to Theorem 6.6, the family $d_V^{\lambda,q}$, on $\lambda\in\mathbb{C}\backslash\{0\}$, consists of a family of flat metric connections on $\underline{\mathbb{C}}^{n+2}$. For non-zero $\lambda\in\mathrm{dom}(r)$ and by definition of $\hat{d}_V^{\lambda,\hat{q}}$, the isometry $r(\lambda)$ is made into an isomorphism

$$
r(\lambda):(\underline{\mathbb{C}}^{n+2}, d_V^{\lambda,q})\to(\underline{\mathbb{C}}^{n+2},\hat{d}_V^{\lambda,\hat{q}}),
$$

ensuring, in particular, that $\hat{d}_V^{\lambda,\hat{q}}$ is a flat connection, as so is $d_V^{\lambda,q}$: the curvature tensors of $\hat{d}_V^{\lambda,\hat{q}}$ and $d_V^{\lambda,q}$ are related by

$$
R^{\hat{d}_V^{\lambda,\hat{q}}} = r(\lambda)\,R^{d_V^{\lambda,q}}\,r(\lambda)^{-1}.
$$

Furthermore, the vanishing of the curvature tensor of $\hat{d}_V^{\lambda,\hat{q}}$ for $\lambda\in\mathrm{dom}(r)\backslash\{0\}$ extends by continuity to $\lambda\in\mathbb{C}\backslash\{0\}$. We are in this way provided with a new $\mathbb{C}\backslash\{0\}$-family of flat metric connections on $\underline{\mathbb{C}}^{n+2}$, that of $\hat{d}_V^{\lambda,\hat{q}}$.

Suppose $1\in\mathrm{dom}(r)$. In the light of Proposition 6.23, and according to Theorem 6.6, the flatness of the metric connection $\hat{d}_V^{\lambda,\hat{q}}$, for $\lambda\in\mathbb{C}\backslash\{0\}$, is equivalent to the $(\hat{q},\hat{d})$-constrained harmonicity of V, which, in its turn and according to Proposition 6.10, is equivalent to the $(\mathrm{Ad}_{r(1)^{-1}}\,\hat{q})$-constrained harmonicity of $r(1)^{-1}V$.

Theorem 6.24. *$r(1)^{-1}V$ is a $(\mathrm{Ad}_{r(1)^{-1}}\,\hat{q})$-constrained harmonic bundle.*

Note that this transformation preserves the harmonicity condition.

This transformation of a constrained harmonic bundle into a new one leads, furthermore, to a transformation of constrained Willmore surfaces into new ones. Suppose, furthermore, that V is a (q,d)-central sphere congruence of some q-constrained Willmore surface $(\Delta^{1,0},\Delta^{0,1})$. Set

$$
\hat{\Delta}^{1,0} := r(\infty)\Delta^{1,0},\quad \hat{\Delta}^{0,1} := r(0)\Delta^{0,1},
$$

and suppose that

$$
\det r(0)_{|V} = \det r(\infty)_{|V}. \tag{6.23}
$$

Then:

Theorem 6.25. *$(r(1)^{-1}\hat{\Delta}^{1,0}, r(1)^{-1}\hat{\Delta}^{0,1})$ is a $(\mathrm{Ad}_{r(1)^{-1}}\hat{q})$-constrained Willmore surface admitting $r(1)^{-1}V$ as a $(\mathrm{Ad}_{r(1)^{-1}}\hat{q}, d)$-central sphere congruence.*

PROOF. The proof will consist of showing that $(\hat{\Delta}^{1,0},\hat{\Delta}^{0,1})$ is a $(\hat{q},\hat{d})$-constrained Willmore surface admitting V as a $(\hat{q},\hat{d})$-central sphere congruence. The result will then follow from Proposition 6.20.

First of all, note that, as q is a multiplier to $(\Delta^{1,0},\Delta^{0,1})$, we have, according to equation (2.18),

$$\hat{q}^{1,0}\in\Omega^{1,0}(\wedge^2\hat{\Delta}^{0,1}),\qquad \hat{q}^{0,1}\in\Omega^{0,1}(\wedge^2\hat{\Delta}^{1,0}).$$

Having observed above that V is $(\hat{q},\hat{d})$-constrained harmonic, it remains to show that $(\hat{\Delta}^{1,0},\hat{\Delta}^{0,1})$ is a $\hat{d}$-surface admitting V as a $\hat{d}$-central sphere congruence, as follows.

The fact $\Delta^{1,0}$ and $\Delta^{0,1}$ are rank 2 isotropic subbundles of V ensures that so are $\Delta^{\hat{1},0}$ and $\Delta^{\hat{0},1}$, as $r(0)$ and $r(\infty)$ are orthogonal transformations and preserve $\Gamma(V)$.

The fact that $\Delta^{1,0}$ and $\Delta^{0,1}$ intersect in a rank 1 bundle ensures that so do $\hat{\Delta}^{1,0}$ and $\hat{\Delta}^{0,1}$. The point is a general fact[2] about the Grassmannian $\mathcal{G}_W$ of isotropic 2-planes in a complex 4-dimensional space W: it has two components, each an orbit of the special orthogonal group $SO(W)$, intertwined by the action of elements of $O(W)\backslash SO(W)$, and for which any element intersects any element of the other component in a line while distinct elements of the same component have trivial intersection. Given that $\operatorname{rank}(\Delta^{1,0}\cap\Delta^{0,1})=1$, $\Delta^{1,0}_p$ and $\Delta^{0,1}_p$ lie in different components of $\mathcal{G}_{V_p}$ and the hypothesis (6.23) ensures that the same is true of $\hat{\Delta}^{1,0}_p$ and $\hat{\Delta}^{0,1}_p$, for all p in M.

Set

$$\hat{\Delta}:=\hat{\Delta}^{1,0}\cap\hat{\Delta}^{0,1}.$$

We are left to verify that

$$\hat{d}^{1,0}\Gamma(\hat{\Delta})\subset\Omega^1(\hat{\Delta}^{1,0}),\quad \hat{d}^{0,1}\Gamma(\hat{\Delta})\subset\Omega^1(\hat{\Delta}^{0,1})\tag{6.24}$$

and that

$$(\mathcal{N}_V^{\hat{d}})^{1,0}\hat{\Delta}^{0,1}=0=(\mathcal{N}_V^{\hat{d}})^{0,1}\hat{\Delta}^{1,0}.\tag{6.25}$$

Equation (6.25) forces $\mathcal{N}_V^{\hat{d}}\hat{\Delta}=0$, in which situation, condition (6.24) reads, equivalently,

$$(\mathcal{D}_V^{\hat{d}})^{1,0}\Gamma(\hat{\Delta})\subset\Omega^1(\hat{\Delta}^{1,0}),\quad (\mathcal{D}_V^{\hat{d}})^{0,1}\Gamma(\hat{\Delta})\subset\Omega^1(\hat{\Delta}^{0,1}),$$

which, in its turn, follows from

$$(\mathcal{D}_V^{\hat{d}})^{1,0}\Gamma(\hat{\Delta}^{1,0})\subset\Omega^1(\hat{\Delta}^{1,0}),\quad (\mathcal{D}_V^{\hat{d}})^{0,1}\Gamma(\hat{\Delta}^{0,1})\subset\Omega^1(\hat{\Delta}^{0,1}).\tag{6.26}$$

It is (6.25) and (6.26) that we shall establish.

By the isotropy of $\Delta^{i,j}$, we have $q^{i,j}\Delta^{i,j}\subset\Delta\subset\Delta^{i,j}$, for $i\neq j\in\{0,1\}$, which, together with Proposition 6.13, makes clear that

$$r(\infty)\cdot(\mathcal{D}_V^{1,0}-q^{1,0})\,\Gamma(\hat{\Delta}^{1,0})\subset\Omega^1(\hat{\Delta}^{1,0}),\ \ r(0)\cdot(\mathcal{D}_V^{0,1}-q^{0,1})\,\Gamma(\hat{\Delta}^{0,1})\subset\Omega^1(\hat{\Delta}^{0,1}).$$

---

[2] That of the two families of lines on the Klein quadric.

On the other hand, as $\hat{\Delta}$ has rank 1, $\hat{\Delta}\wedge\hat{\Delta}^{i,j} = \wedge^2\hat{\Delta}^{i,j}$, so that $\hat{q}^{i,j} \in \Omega^{i,j}(\hat{\Delta}\wedge\hat{\Delta}^{j,i})$, and, therefore, by the isotropy of $\hat{\Delta}^{i,j}$, $\hat{q}^{i,j}\hat{\Delta}^{i,j} \subset \hat{\Delta} \subset \hat{\Delta}^{i,j}$, for $i \neq j \in \{0,1\}$, completing the verification of (6.26), in view of (6.20).

Finally, we establish (6.25). According to (6.21),

$$(\mathcal{N}_V^{\hat{d}})^{1,0} = \mathrm{Ad}_{r(0)}(\mathcal{N}^{1,0} + [r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda), q^{1,0}]).$$

The centrality of V with respect to $(\Delta^{1,0}, \Delta^{0,1})$ establishes, in particular, $\mathcal{N}_V^{1,0}\Delta^{0,1} = 0$, whilst the isotropy of $\Delta^{0,1}$ ensures, in particular, that $q^{1,0}\Delta^{0,1} = 0$. Hence

$$\mathrm{Ad}_{r(0)}(\mathcal{N}^{1,0} + r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda)\, q^{1,0})\hat{\Delta}^{0,1} = 0.$$

On the other hand, differentiation of $r(\lambda)^{-1} = \rho\, r(-\lambda)^{-1}\rho$, derived from equation (6.16), gives

$$-r(\lambda)^{-1}\frac{d}{dk}_{|_{k=\lambda}} r(k)\, r(\lambda)^{-1} = \rho\, r(-\lambda)^{-1}\frac{d}{dk}_{|_{k=-\lambda}} r(k)\, r(-\lambda)^{-1}\rho,$$

or, equivalently,

$$\rho\, r(\lambda)^{-1}\frac{d}{dk}_{|_{k=\lambda}} r(k)\rho = -r(-\lambda)^{-1}\frac{d}{dk}_{|_{k=-\lambda}} r(k)\, r(-\lambda)^{-1}\rho\, r(\lambda)\rho,$$

and, therefore, yet again by equation (6.16),

$$\rho\, r(\lambda)^{-1}\frac{d}{dk}_{|_{k=\lambda}} r(k)\rho = -r(-\lambda)^{-1}\frac{d}{dk}_{|_{k=-\lambda}} r(k).$$

Evaluation at $\lambda = 0$ shows then that

$$\rho\, r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda)\rho = -r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda).$$

Equivalently,

$$r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda) \in \Gamma(V \wedge V^{\perp}). \tag{6.27}$$

As $q^{1,0}V^{\perp} = 0$, we conclude that

$$Ad_{r(0)}(q^{1,0}r(0)^{-1}\frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda))\hat{\Delta}^{0,1} = 0$$

and, ultimately, that $(\mathcal{N}_V^{\hat{d}})^{1,0}\hat{\Delta}^{0,1} = 0$. A similar argument near $\lambda = \infty$ establishes $(\mathcal{N}_V^{\hat{d}})^{0,1}\hat{\Delta}^{1,0} = 0$, completing the proof. □

6.6. Bäcklund transformation of constrained harmonic bundles and complexified constrained Willmore surfaces

The classical Bäcklund transformation was introduced by A. Bäcklund in the nineteenth century, providing a mean to generate constant negative Gaussian curvature

surfaces from a given one. Many variants of this transformation have followed - for details, see [**54**]. In this section, we construct rational maps $r(\lambda)$ satisfying the hypothesis of the dressing action presented above, defining then a transformation of constrained harmonic bundles and complexified constrained Willmore surfaces, the *Bäcklund transformation*. As the philosophy underlying the work of C.-L. Terng and K. Uhlenbeck [**54**] suggests, we consider linear fractional transformations. We define two different types of such transformations, *type* p and *type* q, each one of them satisfying the hypothesis of the dressing action with the exception of condition (0.6). Iterating the procedure, in a 2-step process composing the two different types of transformations, will produce a desired $r(\lambda)$. A *Bianchi permutability* of type p and type q transformations of constrained harmonic bundles is established. For special choices of parameters, the reality of Λ as a bundle proves to establish that of Λ^*, whilst the reality of q establishes that of q^*. For such a choice of parameters, Λ^* is said to be a *Bäcklund transform* of Λ, provided that it immerses.

Let ρ denote reflection across V. Choose $\alpha \in \mathbb{C}\backslash\{-1, 0, 1\}$ and a null line subbundle L of $\underline{\mathbb{C}}^{n+2}$ such that, locally,

$$\rho L \cap L^{\perp} = \{0\}. \tag{6.28}$$

For that, note that condition (6.28), equivalently characterized by $(\rho l, l) \neq 0$, fixing $l \in \Gamma(L)$ never-zero, is an open condition on points of M, so that it is satisfied locally as long as, at some point $p \in M$, L_p is not orthogonal to $\rho_p L_p$, its reflection across V_p. Condition (6.28) ensures, on the one hand, that $L \cap \rho L = \{0\}$, and, on the other hand, that $L \oplus \rho L$ is non-degenerate. Consider then projections $\pi_L : \underline{\mathbb{C}}^{n+2} \to L$, $\pi_{\rho L} : \underline{\mathbb{C}}^{n+2} \to \rho L$ and $\pi_{(L\oplus\rho L)^{\perp}} : \underline{\mathbb{C}}^{n+2} \to (L \oplus \rho L)^{\perp}$ with respect to the decomposition

$$\underline{\mathbb{C}}^{n+2} = L \oplus \rho L \oplus (L \oplus \rho L)^{\perp}.$$

For $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$, set

$$p_{\alpha,L}(\lambda) := \frac{\alpha - \lambda}{\alpha + \lambda}\, \pi_L + \pi_{(L\oplus\rho L)^{\perp}} + \frac{\alpha + \lambda}{\alpha - \lambda}\, \pi_{\rho L}$$

and

$$q_{\alpha,L}(\lambda) := \frac{\lambda - \alpha}{\lambda + \alpha}\, \pi_L + \pi_{(L\oplus\rho L)^{\perp}} + \frac{\lambda + \alpha}{\lambda - \alpha}\, \pi_{\rho L},$$

defining in this way two maps of $\mathbb{C}\backslash\{\pm\alpha\}$ into $\Gamma(O(\underline{\mathbb{C}}^{n+2}))$ that, clearly, extend holomorphically to the Riemann sphere except $\pm\alpha$,

$$p_{\alpha,L}, q_{\alpha,L} : \mathbb{P}^1\backslash\{\pm\alpha\} \to \Gamma(O(\underline{\mathbb{C}}^{n+2})),$$

by setting

$$p_{\alpha,L}(\infty) := -\pi_L + \pi_{(L\oplus\rho L)^{\perp}} - \pi_{\rho L}$$

and

$$q_{\alpha,L}(\infty) := I. \tag{6.29}$$

We may, alternatively, denote $p_{\alpha,L}$ and $q_{\alpha,L}$ by, specifically and respectively, $p_{V,\alpha,L}$ and $q_{V,\alpha,L}$.

The transformations *of type* p and *of type* q are closely related: for $\lambda \in \mathbb{C}\backslash\{\pm\alpha, 0\}$,

$$p_{\alpha,L}(\lambda) = q_{\alpha^{-1},L}(\lambda^{-1}), \tag{6.30}$$

and

$$p_{\alpha,L}(0) = q_{\alpha,L}(\infty), \qquad p_{\alpha,L}(\infty) = q_{\alpha,L}(0). \tag{6.31}$$

Observe that

$$\det p_{\alpha,L}(\infty)_{|V} = \det q_{\alpha,L}(0)_{|V} = -1.$$

In fact,

$$p_{\alpha,L}(\infty)_{|V} = q_{\alpha,L}(0)_{|V} = I \begin{cases} -1 & \text{on } V \cap (L \oplus \rho L) \\ 1 & \text{on } V \cap (L \oplus \rho L)^{\perp} \end{cases}$$

and, as $\rho L \neq L$, L is not a subbundle of V and, therefore, $\operatorname{rank} V \cap (L \oplus \rho L) = 1$ (noting that, for any subbundle W of $\mathbb{R}^{n+1,1}$ and so, in particular, for L, the intersection of $W + \rho W$ with V is not trivial). It follows that $\det p_{\alpha,L}(0)_{|V} \neq \det p_{\alpha,L}(\infty)_{|V}$ and $\det q_{\alpha,L}(0)_{|V} \neq \det q_{\alpha,L}(\infty)_{|V}$; neither $r = p_{\alpha,L}$ nor $r = q_{\alpha,L}$ satisfies the hypothesis (6.23) of the dressing action. However, a two-step process, composing a transformation of *type* p with a transformation of *type* q, produces transformations $r(\lambda)$ satisfying that hypothesis: by (6.29) and (6.31),

$$\det p_{\alpha,L}(0) q_{\alpha',L'}(0)_{|V} = \det p_{\alpha,L}(\infty) q_{\alpha',L'}(\infty)_{|V},$$

as well as

$$\det q_{\alpha,L}(0) p_{\alpha',L'}(0)_{|V} = \det q_{\alpha,L}(\infty) p_{\alpha',L'}(\infty)_{|V},$$

for all α', L'. As we shall verify next, for special choices of parameters α, L, α' and L', both $p_{\alpha,L} q_{\alpha',L'}(\lambda)$ and $q_{\alpha,L} p_{\alpha',L'}(\lambda)$ define $r(\lambda)$ satisfying, furthermore, all the hypotheses of the dressing action.

For that, first note that, for $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha\}$,

$$p_{\alpha,L}(\lambda)^{-1} = p_{-\alpha,L}(\lambda) = p_{\alpha,L}(-\lambda), \quad q_{\alpha,L}(\lambda)^{-1} = q_{-\alpha,L}(\lambda) = q_{\alpha,L}(-\lambda). \tag{6.32}$$

On the other hand, the isometry $\rho = \rho^{-1}$ intertwines L and ρL and, therefore, preserves $(L \oplus \rho L)^{\perp}$, which makes clear that $\rho \circ p_{\alpha,L}$ and $p_{\alpha,L}^{-1} \circ \rho$ coincide in L, ρL and $(L \oplus \rho L)^{\perp}$. Hence

$$\rho\, p_{\alpha,L}(\lambda) \rho = p_{\alpha,L}(-\lambda), \qquad \rho\, q_{\alpha,L}(\lambda) \rho = q_{\alpha,L}(-\lambda), \tag{6.33}$$

for $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha\}$; establishing both $p_{\alpha,L}$ and $q_{\alpha,L}$ - as well as, therefore, $p_{\alpha,L}q_{\alpha',L'}$ and $q_{\alpha,L}p_{\alpha',L'}$, for all α', L' - as twisted in the sense of section 6.5.

Now let q be a 1-form with values in $\wedge^2 V \oplus \wedge^2 V^\perp$. For each $\lambda \in \mathbb{C}\backslash\{-\alpha, 0, \alpha\}$, define a new metric connection on $\underline{\mathbb{C}}^{n+2}$ by setting

$$d^{\lambda,q}_{p_{\alpha,L}} := p_{\alpha,L}(\lambda) \circ d^{\lambda,q}_V \circ p_{\alpha,L}(\lambda)^{-1}.$$

Lemma 6.26. *Suppose L is $d^{\alpha,q}_V$-parallel. In that case, there exists a holomorphic extension of $\lambda \mapsto d^{\lambda,q}_{p_{\alpha,L}}$ to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$.*

Before proceeding to the proof, observe that, as (6.15) makes clear, if L is $d^{\alpha,q}_V$-parallel, then ρL is $d^{-\alpha,q}_V$-parallel.

Proof. Since $d^{\alpha,q}_V$ is a metric connection,

$$d^{\alpha,q}_V\,\Gamma(\rho L) \subset \Omega^1((\rho L)^\perp),$$

as well as, in view of the parallelness of L with respect to $d^{\alpha,q}_V$,

$$d^{\alpha,q}_V\,\Gamma((L\oplus\rho L)^\perp) \subset \Omega^1(L^\perp).$$

For simplicity use $\pi_\perp$ to denote the orthogonal projection of $\underline{\mathbb{C}}^{n+2}$ onto $(L\oplus\rho L)^\perp$. As

$$L^\perp = L \oplus (L\oplus\rho L)^\perp, \quad (\rho L)^\perp = \rho L \oplus (L\oplus \rho L)^\perp,$$

we conclude that $\pi_L \circ d^{\alpha,q}_V \circ \pi_{\rho L} = 0 = \pi_{\rho L} \circ d^{\alpha,q}_V \circ \pi_\perp$, showing that $d^{\alpha,q}_V$ splits as

$$d^{\alpha,q}_V = D^\alpha_q + \beta^\alpha_q$$

for the connection

$$D^\alpha_q := d^{\alpha,q}_V \circ \pi_L + \pi_{\rho L} \circ d^{\alpha,q}_V \circ \pi_{\rho L} + \pi_\perp \circ d^{\alpha,q}_V \circ \pi_\perp,$$

on $\underline{\mathbb{C}}^{n+2}$, and the 1-form

$$\beta^\alpha_q := \pi_\perp \circ d^{\alpha,q}_V \circ \pi_{\rho L} + \pi_L \circ d^{\alpha,q}_V \circ \pi_\perp \in \Omega^1(L \wedge (L\oplus\rho L)^\perp).$$

Clearly, for each λ,

$$p_{\alpha,L}(\lambda) \circ D^\alpha_q \circ p_{\alpha,L}(\lambda)^{-1} = D^\alpha_q, \quad p_{\alpha,L}(\lambda)\,\beta^\alpha_q\,p_{\alpha,L}(\lambda)^{-1} = \frac{\alpha-\lambda}{\alpha+\lambda}\,\beta^\alpha_q.$$

Now decompose $d^{\lambda,q}_V$ as

$$d^{\lambda,q}_V = d^{\alpha,q}_V + (\lambda-\alpha)A(\lambda),$$

for $\lambda \in \mathbb{C}\backslash\{0,\alpha\}$, with $\lambda \mapsto A(\lambda) \in \Omega^1(o(\underline{\mathbb{C}}^{n+2}))$ holomorphic. Namely,

$$A(\lambda) = \frac{\alpha-\lambda}{\alpha\lambda^2 - \alpha^2\lambda}\,\mathcal{N}^{1,0} + \mathcal{N}^{0,1} + \frac{\alpha^2-\lambda^2}{\lambda^3\alpha^2 - \lambda^2\alpha^3}\,q^{1,0} + (\lambda+\alpha)\,q^{0,1},$$

for all λ. It follows that

$$d^{\lambda,q}_{p_{\alpha,L}} = D^\alpha_q + \frac{\alpha-\lambda}{\alpha+\lambda}\,\beta^\alpha_q + (\lambda-\alpha)\,p_{\alpha,L}(\lambda)\,A(\lambda)\,p_{\alpha,L}(\lambda)^{-1},$$

for $\lambda \in \mathbb{C}\backslash\{-\alpha, 0, \alpha\}$. For simplicity, set $\Upsilon(\lambda) := (\lambda - \alpha)\, p_{\alpha,L}(\lambda)\, A(\lambda)\, p_{\alpha,L}(\lambda)^{-1}$. The skew-symmetry of $A(\lambda)$ makes clear that $A(\lambda)L \subset L^{\perp}$, as well as $A(\lambda)\rho L \subset \rho L^{\perp}$ and, consequently, that $\pi_{\rho L}\, A(\lambda)\, \pi_L = 0 = \pi_L\, A(\lambda)\, \pi_{\rho L}$. On the other hand, it is clear that

$$\pi_{\rho L}\, p_{\alpha,L}(\lambda)\, A(\lambda)\, p_{\alpha,L}(\lambda)^{-1}\pi_L = \frac{\alpha+\lambda}{\alpha-\lambda}\, \pi_{\rho L} A(\lambda)\, \frac{\alpha+\lambda}{\alpha-\lambda}\, \pi_L = \frac{(\alpha+\lambda)^2}{(\alpha-\lambda)^2}\, \pi_{\rho L} A(\lambda) \pi_L$$

and, similarly,

$$\pi_L\, p_{\alpha,L}(\lambda)\, A(\lambda)\, p_{\alpha,L}(\lambda)^{-1}\pi_{\rho L} = \frac{(\alpha-\lambda)^2}{(\alpha+\lambda)^2}\, \pi_L\, A(\lambda) \pi_{\rho L}.$$

Hence $\pi_{\rho L}\, \Upsilon(\lambda)\, \pi_L = 0 = \pi_L \Upsilon(\lambda)\, \pi_{\rho L}$. It follows that

$$\begin{aligned} \Upsilon(\lambda) \quad = \quad & (\lambda - \alpha)\, (\pi_L\, A(\lambda)\, \pi_L + \pi_{\rho L}\, A(\lambda)\, \pi_{\rho L} + \pi_{\perp}\, A(\lambda)\, \pi_{\perp}) \\ & - (\alpha + \lambda)\, (\pi_{\perp}\, A(\lambda)\, \pi_L + \pi_{\rho L}\, A(\lambda)\, \pi_{\perp}) \\ & - \frac{(\alpha-\lambda)^2}{\alpha+\lambda}\, (\pi_L\, A(\lambda)\, \pi_{\perp} + \pi_{\perp}\, A(\lambda) \pi_{\rho L}). \end{aligned}$$

Hence, by setting

$$d^{\alpha,q}_{p_{\alpha,L}} := D^{\alpha}_q - 2\alpha\, (\pi_{\perp}\, A(\alpha)\, \pi_L + \pi_{\rho L}\, A(\alpha)\, \pi_{\perp}),$$

we extend holomorphically $\lambda \mapsto d^{\lambda,q}_{p_{\alpha,L}}$ to $\lambda \in \mathbb{C}\backslash\{-\alpha, 0\}$ through what, by continuity, we verify to be a metric connection on $\underline{\mathbb{C}}^{n+2}$.

The existence of a holomorphic extension to $\mathbb{C}\backslash\{0\}$, through a metric connection on $\underline{\mathbb{C}}^{n+2}$, can be proved analogously, having in consideration the $d^{-\alpha,q}_V$-parallelness of ρL. $\square$

The same argument establishes the existence, in the case L is $d^{\alpha,q}_V$-parallel, of a holomorphic extension of

$$\mathbb{C}\backslash\{-\alpha, 0, \alpha\} \ni \lambda \mapsto d^{\lambda,q}_{q_{\alpha,L}} := q_{\alpha,L}(\lambda) \circ d^{\lambda,q}_V \circ q_{\alpha,L}(\lambda)^{-1}$$

to $\mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$. This argument uses nothing about the precise form of the connection $d^{\lambda,q}_V$, only the holomorphicity of $\lambda \mapsto d^{\lambda,q}_V$ in $\mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$, the $d^{\alpha,q}_V$-parallelness of L and the consequent $d^{-\alpha,q}_V$-parallelness of ρL. We can iterate the procedure, in a two-step process, starting with the connections $d^{\lambda,q}_{p_{\alpha,L}}$, defining a family of connections of the form

$$q_{\alpha',L'}(\lambda) p_{\alpha,L}(\lambda) \circ d^{\lambda,q}_V \circ p_{\alpha,L}(\lambda)^{-1} q_{\alpha',L'}(\lambda)^{-1};$$

or, equally, starting with the connections $d^{\lambda,q}_{q_{\alpha,L}}$, defining, in that case, a family of connections of the form

$$p_{\alpha',L'}(\lambda) q_{\alpha,L}(\lambda) \circ d^{\lambda,q}_V \circ q_{\alpha,L}(\lambda)^{-1} p_{\alpha',L'}(\lambda)^{-1};$$

for suitable parameters α, α', L, L', as follows.

Choose L^α a $d^{\alpha,q}_V$-parallel null line subbundle of $\underline{\mathbb{C}}^{n+2}$ with

$$\rho L^\alpha \cap (L^\alpha)^\perp = \{0\}, \tag{6.34}$$

locally. Such L^α can be obtained by choosing a null line $\langle l^\alpha_p\rangle \subset \mathbb{C}^{n+2}$, for some $l^\alpha_p \in \mathbb{C}^{n+2}$ not orthogonal to $\rho_{V_p} l^\alpha_p$, for some $p \in M$, and extending it to a $d^{\alpha,q}_V$-parallel null line subbundle L^α of $\underline{\mathbb{C}}^{n+2}$ by $d^{\alpha,q}_V$-parallel transport of l^α_p. The non-orthogonality of L^α and ρL^α at p is, equivalently, satisfied in some non-empty open set, which we restrict to.

Condition (6.34) allows us to refer to q_{α,L^α}. Now choose $\beta \neq \pm\alpha$ in $\mathbb{C}\backslash\{-1,0,1\}$ and L^β a $d^{\beta,q}_V$-parallel null line subbundle of $\underline{\mathbb{C}}^{n+2}$ and note that the null line bundle

$$\tilde{L}^\beta_\alpha := q_{\alpha,L^\alpha}(\beta)L^\beta$$

is $d^{\beta,q}_{q_{\alpha,L^\alpha}}$-parallel and, consequently, $\rho\tilde{L}^\beta_\alpha$ is parallel with respect to the connection

$$\begin{aligned} d^{-\beta,q}_{q_{\alpha,L^\alpha}} &= q_{\alpha,L^\alpha}(-\beta)\circ d^{-\beta,q}_V \circ \rho\, q_{\alpha,L^\alpha}(\beta)^{-1}\rho \\ &= q_{\alpha,L^\alpha}(-\beta)\rho \circ d^{\beta,q}_V \circ q_{\alpha,L^\alpha}(\beta)^{-1}\rho. \end{aligned}$$

Choose L^β satisfying, furthermore,

$$\rho\tilde{L}^\beta_\alpha \cap (\tilde{L}^\beta_\alpha)^\perp = \{0\}, \tag{6.35}$$

locally. To see that such a choice is possible, first choose a point $p \in M$ at which ρL^α is not orthogonal to L^α. As L^α is an eigenspace of $q_{\alpha,L^\alpha}(\beta)$, $q_{\alpha,L^\alpha}(\beta)L^\alpha = L^\alpha$ and, therefore, at p,

$$\rho q_{\alpha,L^\alpha}(\beta)L^\alpha \cap (q_{\alpha,L^\alpha}(\beta)L^\alpha)^\perp = \{0\}.$$

Choose $l^\alpha_p \in L^\alpha_p$ non-zero. A $d^{\beta,q}_V$-parallel null line bundle $L^\beta \subset \underline{\mathbb{C}}^{n+2}$, satisfying equation (6.35), locally, can be obtained by $d^{\beta,q}_V$-parallel transport of l^α_p.

Condition (6.35) allows us to refer to $p_{\beta,\tilde{L}^\beta_\alpha}$. Set then

$$r^{(\beta,\alpha)}_{L^\alpha,L^\beta} := p_{\beta,\tilde{L}^\beta_\alpha}\, q_{\alpha,L^\alpha},$$

defining, for each $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha,\pm\beta\}$, an orthogonal transformation $r^{(\beta,\alpha)}_{L^\alpha,L^\beta}(\lambda)$ of $\underline{\mathbb{C}}^{n+2}$. The $d^{\alpha,q}_V$-parallelness of L^α ensures that $\lambda \mapsto d^{\lambda,q}_{q_{\alpha,L^\alpha}}$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$ and, consequently, the $d^{\beta,q}_{q_{\alpha,L^\alpha}}$-parallelness of $\tilde{L}^\beta_\alpha$ (together with the consequent $d^{-\beta,q}_{q_{\alpha,L^\alpha}}$-parallelness of $\rho\tilde{L}^\beta_\alpha$) ensures that so does

$$\lambda \mapsto r^{(\beta,\alpha)}_{L^\alpha,L^\beta}(\lambda)\circ d^{\lambda,q}_V \circ r^{(\beta,\alpha)}_{L^\alpha,L^\beta}(\lambda)^{-1} = p_{\beta,\tilde{L}^\beta_\alpha}(\lambda)\, q_{\alpha,L^\alpha}(\lambda)\circ d^{\lambda,q}_V \circ q_{\alpha,L^\alpha}(\lambda)^{-1}\, p_{\beta,\tilde{L}^\beta_\alpha}(\lambda)^{-1}.$$

We conclude that $r^{(\beta,\alpha)}_{L^\alpha,L^\beta}$ satisfies the hypothesis of the dressing action, defining a transformation of constrained harmonic bundles and constrained Willmore surfaces.

Our next step is to investigate how these transformations relate to the ones defined by $r^{(\alpha,\beta)}_{L^\alpha,L^\beta}$, in the case they are both defined.

Suppose, furthermore, that, locally, L^β is never-orthogonal to ρL^β,

$$\rho L^\beta \cap (L^\beta)^\perp = \{0\}. \tag{6.36}$$

This is certainly the case for L^β obtained by $d_V^{\beta,q}$-parallel transport of l^α_p, for $l^\alpha_p \in L^\alpha_p$ non-zero and p a point in M at which ρL^α is not orthogonal to L^α. Note that

$$\tilde{L}^\alpha_\beta = q_{\beta,L^\beta}(\alpha)L^\alpha$$

is a $d^{\alpha,q}_{q_{\beta,L^\beta}}$-parallel bundle. Observe that, locally,

$$\rho\tilde{L}^\alpha_\beta \cap (\tilde{L}^\alpha_\beta)^\perp = \{0\}. \tag{6.37}$$

Indeed, given $l^\alpha \in \Gamma(L^\alpha)$ never-zero, and according to (6.32) and (6.33), we have

$$\begin{aligned}(\rho\, q_{\beta,L^\beta}(\alpha)l^\alpha, q_{\beta,L^\beta}(\alpha)l^\alpha) &= (q_{\beta,L^\alpha}(\alpha)^{-1}\rho\, l^\alpha, q_{\beta,L^\beta}(\alpha)l^\alpha)\\ &= (\rho\, l^\alpha, q_{\beta,L^\alpha}(\alpha)^2 l^\alpha)\\ &= \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\,(\rho\, l^\alpha, l^\alpha);\end{aligned}$$

so that, given p a point in M at which ρL^α is not orthogonal to L^α,

$$(\rho_{V_p}\, q_{\beta,L^\beta_p}(\alpha)l^\alpha_p, q_{\beta,L^\beta_p}(\alpha)l^\alpha_p) \neq 0,$$

condition (6.37) is satisfied at p, or, equivalently, in some open neighbourhood of p.

Set then

$$r^{(\alpha,\beta)}_{L^\alpha,L^\beta} := p_{\alpha,\tilde{L}^\alpha_\beta}\, q_{\beta,L^\beta},$$

defining, for each $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha,\pm\beta\}$, an orthogonal transformation $r^{(\alpha,\beta)}_{L^\alpha,L^\beta}(\lambda)$ of $\underline{\mathbb{C}}^{n+2}$. The $d_V^{\beta,q}$-parallelness of L^β ensures that $\lambda \mapsto d^{\lambda,q}_{q_{\beta,L^\beta}}$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$ and, consequently, the $d^{\alpha,q}_{q_{\beta,L^\beta}}$-parallelness of $\tilde{L}^\alpha_\beta$ (together with the consequent $d^{-\alpha,q}_{q_{\beta,L^\beta}}$-parallelness of $\rho\tilde{L}^\alpha_\beta$) ensures that so does

$$\lambda \mapsto r^{(\alpha,\beta)}_{L^\alpha,L^\beta}(\lambda)\circ d_V^{\lambda,q}\circ r^{(\alpha,\beta)}_{L^\alpha,L^\beta}(\lambda)^{-1} = p_{\alpha,\tilde{L}^\alpha_\beta}(\lambda)\, q_{\beta,L^\beta}(\lambda)\circ d_V^{\lambda,q}\circ q_{\beta,L^\beta}(\lambda)^{-1}p_{\alpha,\tilde{L}^\alpha_\beta}(\lambda)^{-1}.$$

We conclude that $r^{(\alpha,\beta)}_{L^\alpha,L^\beta}$ satisfies the hypothesis of the dressing action, defining a transformation of constrained harmonic bundles and constrained Willmore surfaces. We shall verify that these transformations coincide with the ones defined by $r^{(\beta,\alpha)}_{L^\alpha,L^\beta}$. On the way, we verify that, starting with the connections $d^{\lambda,q}_{p_{\alpha,L^\alpha}}$, rather than with the connections $d^{\lambda,q}_{q_{\beta,L^\beta}}$, leads to the same transformations of constrained harmonic bundles and of constrained Willmore surfaces.

Set

$$\hat{L}^{\beta}_{\alpha} := p_{\alpha,L^{\alpha}}(\beta)L^{\beta}$$

and observe that, locally,

$$\rho\hat{L}^{\beta}_{\alpha} \cap (\hat{L}^{\beta}_{\alpha})^{\perp} = \{0\}.$$

Indeed, given $l^{\beta} \in \Gamma(L^{\beta})$,

$$(\rho p_{\alpha,L^{\alpha}}(\beta)l^{\beta}, p_{\alpha,L^{\alpha}}(\beta)l^{\beta}) = \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\,(\rho\, l^{\beta}, l^{\beta}),$$

and the conclusion follows, in view of (6.36).

Set then

$$\hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}} := q_{\beta,\hat{L}^{\beta}_{\alpha}}\, p_{\alpha,L^{\alpha}},$$

defining, for each $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha,\pm\beta\}$, an orthogonal transformation $\hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}(\lambda)$ of $\underline{\mathbb{C}}^{n+2}$. The $d^{\alpha,q}_{V}$-parallelness of L^{α} ensures that $\lambda \mapsto d^{\lambda,q}_{p_{\alpha,L^{\alpha}}}$ admits a holomorphic extension to $\lambda \in \mathbb{C}\backslash\{0\}$ through metric connections on $\underline{\mathbb{C}}^{n+2}$ and, consequently, the $d^{\beta,q}_{p_{\alpha,L^{\alpha}}}$-parallelness of $\hat{L}^{\beta}_{\alpha}$ (and the consequent $d^{-\beta,q}_{p_{\alpha,L^{\alpha}}}$-parallelness of $\rho\hat{L}^{\beta}_{\alpha}$) ensures that so does

$$\lambda \mapsto \hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}(\lambda) \circ d^{\lambda,q}_{V} \circ \hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}(\lambda)^{-1} = q_{\beta,\hat{L}^{\beta}_{\alpha}}(\lambda)p_{\alpha,L^{\alpha}}(\lambda) \circ d^{\lambda,q}_{V} \circ p_{\alpha,L^{\alpha}}(\lambda)^{-1}q_{\beta,\hat{L}^{\beta}_{\alpha}}(\lambda)^{-1}.$$

We conclude that $\hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}$ satisfies the hypothesis of the dressing action, defining a transformation of constrained harmonic bundles and constrained Willmore surfaces.

For simplicity, set $r^* := r^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}$ and $\hat{r}^* := \hat{r}^{(\alpha,\beta)}_{L^{\alpha},L^{\beta}}$.

Proposition 6.27. *r^* and $\hat{r}^*$ are related by*

$$r^* = K\,\hat{r}^*, \tag{6.38}$$

for $K := q_{\beta,L^{\beta}}(0)\, q_{\beta,\hat{L}^{\beta}_{\alpha}}(0)$.

The proof of the proposition will be based on the following lemma, cf. F. Burstall [**10**].

Lemma 6.28. *Let $\gamma(\lambda) = \lambda\,\pi_{L_1} + \pi_{L_0} + \lambda^{-1}\,\pi_{L_{-1}}$ and $\hat{\gamma}(\lambda) = \lambda\,\pi_{\hat{L}_1} + \pi_{\hat{L}_0} + \lambda^{-1}\,\pi_{\hat{L}_{-1}}$ be homomorphisms of $\mathbb{C}^{n+2}$ corresponding to decompositions*

$$\mathbb{C}^{n+2} = L_1 \oplus L_0 \oplus L_{-1} = \hat{L}_1 \oplus \hat{L}_0 \oplus \hat{L}_{-1}$$

with $L_{\pm 1}$ and $\hat{L}_{\pm 1}$ null lines and $L_0 = (L_1 \oplus L_{-1})^{\perp}$, $\hat{L}_0 = (\hat{L}_1 \oplus \hat{L}_{-1})^{\perp}$. Suppose $\mathrm{Ad}\,\gamma$ and $\mathrm{Ad}\,\hat{\gamma}$ have simple poles. Suppose as well that ξ is a map into $O(\mathbb{C}^{n+2})$ holomorphic near 0 such that

$$L_1 = \xi(0)\hat{L}_1.$$

Then $\gamma\xi\hat{\gamma}^{-1}$ is holomorphic and invertible at 0.

Now we proceed to the proof of Proposition 6.27.

Proof. For simplicity, throughout this proof, we adopt $p_{\mu,L}^{-1}$ and $q_{\mu,L}^{-1}$ to denote $\lambda \mapsto p_{\mu,L}(\lambda)^{-1}$ and, respectively, $\lambda \mapsto q_{\mu,L}(\lambda)^{-1}$, in the case $p_{\mu,L}$ and, respectively, $q_{\mu,L}$ are defined.

As $L^\alpha = q_{\beta,L^\beta}(\alpha)^{-1}\tilde{L}_\beta^\alpha$, after an appropriate change of variable, we conclude, by Lemma 6.28, that $p_{\alpha,L^\alpha}\, q_{\beta,L^\beta}^{-1}\, p_{\alpha,\tilde{L}_\beta^\alpha}^{-1}$ admits a holomorphic and invertible extension to $\mathbb{P}^1\backslash\{\pm\beta,-\alpha\}$. On the other hand, in view of (6.33), the holomorphicity and invertibility of $p_{\alpha,L^\alpha}\, q_{\beta,L^\beta}^{-1}\, p_{\alpha,\tilde{L}_\beta^\alpha}^{-1}$ at the points α and $-\alpha$ are equivalent. Thus $p_{\alpha,L^\alpha}\, q_{\beta,L^\beta}^{-1}\, p_{\alpha,\tilde{L}_\beta^\alpha}^{-1}$ admits an holomorphic and invertible extension to $\mathbb{P}^1\backslash\{\pm\beta\}$, and so does, therefore, $(p_{\alpha,L^\alpha}\, q_{\beta,L^\beta}^{-1}\, p_{\alpha,\tilde{L}_\beta^\alpha}^{-1})^{-1}\, q_{\beta,L^\beta}^{-1}$. A similar argument shows that $p_{\alpha,\tilde{L}_\beta^\alpha}\,(q_{\beta,L^\beta}\, p_{\alpha,L^\alpha}^{-1}\, q_{\beta,\hat{L}_\alpha^\beta}^{-1})$ admits an holomorphic extension to $\mathbb{P}^1\backslash\{\pm\alpha\}$. But

$$p_{\alpha,\tilde{L}_\beta^\alpha}\, q_{\beta,L^\beta}\, p_{\alpha,L^\alpha}^{-1}\, q_{\beta,\hat{L}_\alpha^\beta}^{-1} = (p_{\alpha,L^\alpha}\, q_{\beta,L^\beta}^{-1}\, p_{\alpha,\tilde{L}_\beta^\alpha}^{-1})^{-1}\, q_{\beta,L^\beta}^{-1}.$$

We conclude that $p_{\alpha,\tilde{L}_\beta^\alpha}\, q_{\beta,L^\beta}\, p_{\alpha,L^\alpha}^{-1}\, q_{\beta,\hat{L}_\alpha^\beta}^{-1}$ extends holomorphically to $\mathbb{P}^1$ and is, therefore, constant. Evaluating at $\lambda = 0$ gives $p_{\alpha,\tilde{L}_\beta^\alpha}\, q_{\beta,L^\beta}\, p_{\alpha,L^\alpha}^{-1}\, q_{\beta,\hat{L}_\alpha^\beta}^{-1} = q_{\beta,L^\beta}(0)\, q_{\beta,\hat{L}_\alpha^\beta}(0)$, completing the proof. □

According to (6.33), K commutes with ρ,

$$\rho\, K\, \rho = K. \tag{6.39}$$

This ensures, in particular, that K preserves V: given $v \in \Gamma(V)$, $\rho K v = \rho K \rho v = K v$. Equivalently,

$$K\, V = V. \tag{6.40}$$

Together with equation (6.38), equation (6.40) shows that, for each λ,

$$\hat{r}^{\,*}(\lambda)^{-1}\, V = r^*(\lambda)^{-1}\, K\, V = r^*(\lambda)^{-1}\, V.$$

In particular,

$$\hat{r}^{\,*}(1)^{-1}\, V = r^*(1)^{-1}\, V. \tag{6.41}$$

As for a constrained Willmore surface $(\Delta^{1,0}, \Delta^{0,1})$ admitting V as a (q,d)-central sphere congruence, and, yet again, by equation (6.38), we have

$$r^*(1)^{-1} r^*(\infty)\Delta^{1,0} = \hat{r}^{\,*}(1)^{-1}\, K^{-1}\,(K\,\hat{r}^{\,*})(\infty)\,\Delta^{1,0} = \hat{r}^{\,*}(1)^{-1}\,\hat{r}^{\,*}(\infty)\,\Delta^{1,0},$$

as well as

$$r^*(1)^{-1} r^*(0)\Delta^{0,1} = \hat{r}^{\,*}(1)^{-1}\,\hat{r}^{\,*}(0)\,\Delta^{0,1}.$$

We conclude that, despite not coinciding, r^* and $\hat{r}^{\,*}$ produce the same transformations of bundles and complexified surfaces.

Now set

$$\tilde{K} = p_{\alpha,L^\alpha}(\infty)\, p_{\beta,L^\beta}(\infty).$$

Note that

$$p_{\alpha',L'}(\lambda)\,p_{\alpha',L'}(\infty)=q_{\alpha',L'}(\lambda)=p_{\alpha',L'}(\infty)\,p_{\alpha',L'}(\lambda),$$

or, equivalently,

$$q_{\alpha',L'}(\lambda)\,p_{\alpha',L'}(\infty)=p_{\alpha',L'}(\lambda)=p_{\alpha',L'}(\infty)\,q_{\alpha',L'}(\lambda),$$

for all α', L' and $\lambda\in\mathbb{P}^1\backslash\{\pm\alpha'\}$. Together with equation (6.38), and having in consideration that $p_{\alpha',L'}(\infty)^2=I$, this establishes

$$\begin{aligned}\tilde{K}\,r^* &= p_{\alpha,L^\alpha}(\infty)\,p_{\beta,L^\beta}(\infty)^2\,p_{\beta,\hat{L}^\beta_\alpha}(\infty)\,q_{\beta,\hat{L}^\beta_\alpha}p_{\alpha,L^\alpha}\\ &= p_{\alpha,L^\alpha}(\infty)\,p_{\beta,\hat{L}^\beta_\alpha}\,p_{\alpha,L^\alpha}\\ &= p_{\beta,\hat{L}^\beta_\alpha}\,p_{\alpha,L^\alpha}(\infty)p_{\alpha,L^\alpha}\end{aligned}$$

and, ultimately,

$$r^{(\beta,\alpha)}_{L^\alpha,L^\beta}=\tilde{K}\,r^{(\alpha,\beta)}_{L^\alpha,L^\beta}. \tag{6.42}$$

Ultimately, we conclude that, despite not coinciding, $r^{(\alpha,\beta)}_{L^\alpha,L^\beta}$ and $r^{(\beta,\alpha)}_{L^\alpha,L^\beta}$ produce the same transformations of bundles and complexified surfaces.

Set

$$q^*:=\mathrm{Ad}_{r^*(1)^{-1}}(\mathrm{Ad}_{r^*(0)}q^{1,0}+\mathrm{Ad}_{r^*(\infty)}q^{0,1}).$$

Definition 6.29. *Suppose V is a q-constrained harmonic bundle. In that case, the q^*-constrained harmonic bundle*

$$V^*:=r^*(1)^{-1}\,V$$

is said to be the Bäcklund transform of V of parameters $\alpha,\beta,L^\alpha,L^\beta$. *In the case V is a (q,d)-central sphere congruence for some constrained Willmore surface $(\Delta^{1,0},\Delta^{0,1})$, the q^*-constrained Willmore surface*

$$(\Delta^{1,0},\,\Delta^{0,1})^*:=((\Delta^*)^{1,0},\,(\Delta^*)^{0,1}):=(r^*(1)^{-1}\,r^*(\infty)\Delta^{1,0},r^*(1)^{-1}\,r^*(0)\Delta^{0,1})$$

is said to be the Bäcklund transform of $(\Delta^{1,0},\Delta^{0,1})$ of parameters $\alpha,\beta,L^\alpha,L^\beta$.

Remark 6.30. *Set $\tilde{r}^*:=r^{(\beta,\alpha)}_{L^\alpha,L^\beta}$. According to equations* (6.38) *and* (6.42)*,*

$$\hat{q}^*:=\mathrm{Ad}_{\hat{r}^*(1)^{-1}}(\mathrm{Ad}_{\hat{r}^*(0)}q^{1,0}+\mathrm{Ad}_{\hat{r}^*(\infty)}q^{0,1})=q^*,$$

as well as

$$\tilde{q}^*:=\mathrm{Ad}_{\tilde{r}^*(1)^{-1}}(\mathrm{Ad}_{\tilde{r}^*(0)}q^{1,0}+\mathrm{Ad}_{\tilde{r}^*(\infty)}q^{0,1})=q^*.$$

Equation (6.38) establishes a *Bianchi permutability*[3] of type p and type q transformations of constrained harmonic bundles (into constrained harmonic bundles), by means of the commutativity of the diagram in Figure 6-1, below.

[3]The terminology is motivated by the permutability of this kind established by Bianchi with respect to the original Bäcklund transformations.

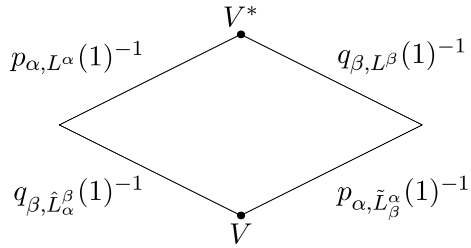

FIGURE 6-1. A Bianchi permutability of type p and type q transformations.

Equation (6.38) will play a crucial role when investigating the preservation of reality conditions by Bäcklund transformations, in the next section.

6.6.1. Real Bäcklund transformation. As we verify in this section, for special choices of parameters, the Bäcklund transformation preserves reality conditions.

Suppose V is a real q-constrained harmonic subbundle of $\underline{\mathbb{C}}^{n+2}$ and let us focus on the particular case of a Bäcklund transformation of parameters $\alpha, \beta, L^\alpha, L^\beta$ for

$$\alpha \in \mathbb{C}\backslash(S^1 \cup \{0\}), \qquad \beta = \overline{\alpha}^{\,-1}, \qquad L^\beta = \overline{L^\alpha}$$

and L^α a $d_V^{\alpha,q}$-parallel null line subbundle of $\underline{\mathbb{C}}^{n+2}$ such that, locally,

$$\rho L^\alpha \cap (L^\alpha)^\perp = \{0\}$$

and

$$\rho\, q_{\overline{\alpha}^{\,-1},\overline{L^\alpha}}(\alpha) L^\alpha \cap (q_{\overline{\alpha}^{\,-1},\overline{L^\alpha}}(\alpha) L^\alpha)^\perp = \{0\}. \tag{6.43}$$

We start by verifying that this is, indeed, a possible choice of Bäcklund transformation parameters in the case V is real.

First note that the reality of V establishes that of ρ, $\overline{\rho} = \rho$, in which case the local non-orthogonality of L^α and ρL^α is equivalent to that of $\overline{L^\alpha}$ and $\rho \overline{L^\alpha}$,

$$\rho \overline{L^\alpha} \cap (\overline{L^\alpha})^\perp = \{0\}.$$

On the other hand, as V and $V^\perp$ are real, so are then π_V and $\pi_{V^\perp}$, as well as, therefore, $\mathcal{D}_V$ and $\mathcal{N}_V$, so that, by the reality of q,

$$d_V^{\overline{\alpha}^{\,-1},q} = \overline{d_V^{\alpha,q}}.$$

Hence the $d_V^{\alpha,q}$-parallelness of L^α is equivalent to the $d_V^{\overline{\alpha}^{\,-1},q}$-parallelness of $\overline{L^\alpha}$. The reality of ρ establishes, on the other hand,

$$\overline{p_{V,\alpha',L'}(\lambda)} = p_{\overline{V},\overline{\alpha'},\overline{L'}}(\overline{\lambda}), \qquad \overline{q_{V,\alpha',L'}(\lambda)} = q_{\overline{V},\overline{\alpha'},\overline{L'}}(\overline{\lambda}), \tag{6.44}$$

together with

$$\overline{p_{V,\alpha',L'}(\infty)} = p_{\overline{V},\overline{\alpha'},\overline{L'}}(\infty),$$

for all α', L' and $\lambda \in \mathbb{C}\backslash\{\pm\alpha', 0\}$. According to equation (6.30), it follows that, given $l^\alpha \in \Gamma(L^\alpha)$,

$$\begin{aligned}\overline{(\rho q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha) l^\alpha, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha) l^\alpha)} &= (\rho q_{\alpha^{-1},L^\alpha}(\overline{\alpha})\overline{l^\alpha}, q_{\alpha^{-1},L^\alpha}(\overline{\alpha})\overline{l^\alpha}) \\ &= (\rho p_{\alpha,L^\alpha}(\overline{\alpha}^{-1})\overline{l^\alpha}, p_{\alpha,L^\alpha}(\overline{\alpha}^{-1})\overline{l^\alpha}),\end{aligned}$$

showing that condition (6.43) is equivalent to

$$\rho p_{\alpha,L^\alpha}(\overline{\alpha}^{-1})\overline{L^\alpha} \cap (p_{\alpha,L^\alpha}(\overline{\alpha}^{-1})\overline{L^\alpha})^\perp = \{0\}.$$

Next we establish the existence of a choice of L^α in the conditions above.

Lemma 6.31. *Let v and w be sections of V and $V^\perp$, respectively, with (v,v) never-zero, $(v,\overline{v}) = 0$ and $(w,w) = -(v,v)$. Define a null section of $\underline{\mathbb{C}}^{n+2}$ by $l^\alpha := v + w$. Let $L^\alpha \subset \underline{\mathbb{C}}^{n+2}$ be a $d_V^{\alpha,q}$-parallel null line bundle defined naturally by $d_V^{\alpha,q}$-parallel transport of l^α_p, for some point $p \in M$. Then there is some (non-empty) open set in which L^α is never orthogonal to ρL^α and satisfies equation* (6.43).

Proof. At the point p, L^α is spanned by l^α_p. The fact that, at the point p,

$$(\rho l^\alpha, l^\alpha) = (v,v) - (w,w) = 2(v,v) \neq 0$$

establishes the non-orthogonality of L^α and ρL^α at this point and, therefore, locally. Consider projections $\pi_{\overline{L^\alpha}} : \underline{\mathbb{C}}^{n+2} \to \overline{L^\alpha}$, $\pi_{\rho\overline{L^\alpha}} : \underline{\mathbb{C}}^{n+2} \to \rho\overline{L^\alpha}$ and $\pi_\perp : \underline{\mathbb{C}}^{n+2} \to (\overline{L^\alpha} \oplus \rho\overline{L^\alpha})^\perp$ with respect to the decomposition

$$\underline{\mathbb{C}}^{n+2} = \overline{L^\alpha} \oplus (\overline{L^\alpha} \oplus \rho\overline{L^\alpha})^\perp \oplus \rho\overline{L^\alpha},$$

established by the subsequent local non-orthogonality of $\overline{L^\alpha}$ and $\rho\overline{L^\alpha}$. For simplicity, denote $q_{\overline{\alpha}^{-1},\overline{L^\alpha}}$ by q. Set

$$A := \frac{\alpha - \overline{\alpha}^{-1}}{\alpha + \overline{\alpha}^{-1}} = \frac{|\alpha|^2 - 1}{|\alpha|^2 + 1} \in \mathbb{R}.$$

Then

$$\rho q(\alpha) l^\alpha = q(\alpha)^{-1} \rho l^\alpha = A^{-1} \pi_{\overline{L^\alpha}} \rho l^\alpha + \pi_\perp \rho l^\alpha + A\, \pi_{\rho\overline{L^\alpha}} \rho l^\alpha,$$

and, therefore,

$$(\rho q(\alpha) l^\alpha, q(\alpha) l^\alpha) = A^2 (\pi_{\overline{L^\alpha}} l^\alpha, \pi_{\rho\overline{L^\alpha}} \rho l^\alpha) + (\pi_\perp l^\alpha, \pi_\perp \rho l^\alpha) + A^{-2} (\pi_{\rho\overline{L^\alpha}} l^\alpha, \pi_{\overline{L^\alpha}} \rho l^\alpha).$$

At the point p, $\overline{L^\alpha} = \langle \overline{l^\alpha} \rangle$ and the orthogonality relations show then that, at p,

$$l^\alpha = \frac{(l^\alpha, \rho\overline{l^\alpha})}{(\rho l^\alpha, l^\alpha)} \overline{l^\alpha} + \frac{(l^\alpha, \overline{l^\alpha})}{(\rho l^\alpha, l^\alpha)} \rho\overline{l^\alpha} + \pi_\perp l^\alpha.$$

Hence, at p,

$$(\pi_{\overline{L^\alpha}}\, l^\alpha, \pi_{\rho\overline{L^\alpha}}\, \rho l^\alpha) = \frac{(l^\alpha, \rho\overline{l^\alpha})^2}{\overline{(\rho l^\alpha, l^\alpha)}}$$

and

$$(\pi_{\rho\overline{L^\alpha}}\, l^\alpha, \pi_{\overline{L^\alpha}}\, \rho l^\alpha) = \frac{(l^\alpha, \overline{l^\alpha})^2}{\overline{(\rho l^\alpha, l^\alpha)}},$$

and, therefore,

$$(\pi_\perp l^\alpha, \pi_\perp \rho l^\alpha) \quad = \quad (\rho l^\alpha, l^\alpha) - \frac{(l^\alpha, \rho\overline{l^\alpha})^2 + (l^\alpha, \overline{l^\alpha})^2}{\overline{(\rho l^\alpha, l^\alpha)}}.$$

It follows that $(\rho q(\alpha) l^\alpha, q(\alpha) l^\alpha)$ vanishes at p if and only if, at this point,

$$\mid (\rho l^\alpha, l^\alpha) \mid^2 + (A^2 - 1)(l^\alpha, \rho\overline{l^\alpha})^2 + (A^{-2} - 1)(l^\alpha, \overline{l^\alpha})^2 = 0,$$

or, equivalently,

$$\mid (\rho l^\alpha, l^\alpha) \mid^2 + (A^2 + A^{-2} - 2)(l^\alpha, \overline{l^\alpha})^2 = 0,$$

as, since $(v, \overline{v}) = 0$, we have, at p,

$$(l^\alpha, \rho\overline{l^\alpha})^2 = ((v, \overline{v}) - (w, \overline{w}))^2 = (w, \overline{w})^2 = ((v, \overline{v}) + (w, \overline{w}))^2 = (l^\alpha, \overline{l^\alpha})^2.$$

As $(l^\alpha, \overline{l^\alpha})$ is real, and, therefore, $(l^\alpha, \overline{l^\alpha})^2 \geq 0$, and $A^2 + A^{-2} - 2 = (A - A^{-1})^2 > 0$, we conclude that, at the point p,

$$(\rho q(\alpha) l^\alpha, q(\alpha) l^\alpha) \neq 0.$$

The proof is complete by the fact that the non-orthogonality of $q(\alpha) L^\alpha$ and $\rho q(\alpha) L^\alpha$ is an open condition on the points in M. □

We refer to a Bäcklund transformation corresponding to this particular choice of parameters, in the case V is real, as a *Bäcklund transformation of parameters* α, L^α. For this choice of parameters, where, in particular, β and L^β are defined by α and L^α, we denote $\tilde{L}^\alpha_\beta$, $\hat{L}^\beta_\alpha$ and $r^{(\alpha,\beta)}_{L^\alpha, L^\beta}$ simply by $\tilde{L}^\alpha$, $\hat{L}^{\overline{\alpha}^{-1}}$ and $r^\alpha_{L^\alpha}$, respectively. In what follows in this section, consider this particular choice of parameters. Let r^* denote $r^\alpha_{L^\alpha}$. According to (6.30) and (6.44),

$$\begin{aligned} \overline{r^*(1)^{-1}} \quad &= \quad \overline{q_{\overline{\alpha}^{-1}, \overline{L^\alpha}}(1)}^{\,-1}\, \overline{p_{\alpha, \tilde{L}^\alpha}(1)}^{\,-1} \\ &= \quad p_{\alpha, L^\alpha}(1)^{\,-1}\, p_{\overline{\alpha}, \overline{\tilde{L}^\alpha}}(1)^{\,-1}. \end{aligned}$$

On the other hand,

$$\overline{\tilde{L}^\alpha} = q_{\alpha^{-1}, L^\alpha}(\overline{\alpha})\overline{L^\alpha} = p_{\alpha, L^\alpha}(\overline{\alpha}^{\,-1})\overline{L^\alpha} = \hat{L}^{\overline{\alpha}^{\,-1}}. \tag{6.45}$$

and, therefore, by equation (6.38),

$$\begin{aligned} r^*(1)^{-1} \quad &= \quad (K q_{\overline{\alpha}^{\,-1}, \overline{\tilde{L}^\alpha}}(1)\, p_{\alpha, L^\alpha}(1))^{-1} \\ &= \quad p_{\alpha, L^\alpha}(1)^{-1}\, p_{\overline{\alpha}, \overline{\tilde{L}^\alpha}}(1)^{-1}\, K^{-1}. \end{aligned}$$

Hence

$$\overline{r^*(1)^{-1}} = r^*(1)^{-1}K. \tag{6.46}$$

Note that, given α', L', $p_{\alpha',L'}(\infty)$ does not depend on α'. Thus

$$\begin{aligned}\overline{r^*(0)} &= \overline{p_{\alpha,\tilde{L}^\alpha}(0)q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(0)}\\ &= p_{\alpha^{-1},L^\alpha}(\infty)\\ &= p_{\alpha,L^\alpha}(\infty).\end{aligned}$$

On the other hand, by equation (6.38),

$$\begin{aligned}r^*(\infty) &= Kq_{\overline{\alpha}^{-1},\hat{L}^{\overline{\alpha}^{-1}}}(\infty)p_{\alpha,L^\alpha}(\infty)\\ &= Kp_{\alpha,L^\alpha}(\infty).\end{aligned}$$

We conclude that

$$\overline{r^*(0)} = K^{-1}r^*(\infty). \tag{6.47}$$

Remark 6.32. *It is opportune to remark that, by* (6.45)*,*

$$\begin{aligned}\overline{r^*(\lambda)} &= \overline{K\,q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\lambda)\,p_{\alpha,L^\alpha}(\lambda)} =\\ &= \overline{K}\,p_{\alpha,\tilde{L}^\alpha}(\overline{\lambda}^{-1})\,q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\overline{\lambda}^{-1})\\ &= \overline{K}\,r^*(\overline{\lambda}^{-1})\end{aligned}$$

for all λ, and, in particular,

$$\overline{r^*(1)^{-1}} = r^*(1)^{-1}\overline{K}^{-1}. \tag{6.48}$$

Together, equations (6.46) *and* (6.48) *establish, in particular, $\overline{K} = K^{-1}$. On the other hand, in view of the fact that given α', L', $p_{\alpha',L'}(\infty)$ does not depend on α',* (6.32) *establishes $K^{-1} = K$. Hence*

$$\overline{K} = K^{-1} = K. \tag{6.49}$$

Let V^* be the Bäcklund transform of V of parameters α, L^α. Equation (6.46), together with equation (6.40), shows that the reality of V establishes that of V^*,

$$\overline{V^*} = V^*.$$

Following Theorem 6.24, we get:

Theorem 6.33. *If V is a real q-constrained harmonic bundle, then the Bäcklund transform V^* of V, of parameters α, L^α, is a real q^*-constrained harmonic bundle.*

Bäcklund transformations of constrained harmonic bundles preserve reality conditions, for this specific choice of parameters. As we verify next, this leads us, via the

central sphere congruence, to a transformation of real constrained Willmore surfaces, preserving the Willmore surface condition.

Suppose Λ is a real q-constrained Willmore surface having V as the complexification of its central sphere congruence. In particular, q is real. According to equations (6.46) and (6.47), the reality of q establishes that of q^*:

$$\begin{aligned}\overline{(q^*)^{1,0}} &= \overline{r^*(1)^{-1}}\,\overline{r^*(0)}\,\overline{q^{1,0}}\,\overline{r^*(1)^{-1}\,r^*(0)}^{\,-1}\\ &= r^*(1)^{-1}r^*(\infty)\,q^{0,1}\,r^*(\infty)^{-1}\,r^*(1)\\ &= (q^*)^{0,1}\end{aligned}$$

and, therefore,

$$\overline{q^*} = q^*.$$

Let $((\Lambda^*)^{1,0},(\Lambda^*)^{0,1})$ be the Bäcklund transform of Λ of parameters α, L^α. Yet again according to equations (6.46) and (6.47),

$$\begin{aligned}\overline{(\Lambda^*)^{0,1}} &= \overline{r^*(1)^{-1}}\,\overline{r^*(0)}\,\overline{\Lambda^{0,1}}\\ &= r^*(1)^{-1}\,r^*(\infty)\Lambda^{1,0}\\ &= (\Lambda^*)^{1,0}\end{aligned}$$

establishing the reality of the bundle

$$\Lambda^* := (\Lambda^*)^{1,0}\cap(\Lambda^*)^{0,1}.$$

If Λ^* defines an immersion of M into $\mathbb{P}(\mathcal{L})$, then, according to Theorem 6.25, V^* is the complexification of the central sphere congruence of Λ^*, which establishes Λ^* as a real q^*-constrained Willmore surface.

Theorem 6.34. *If Λ is a real q-constrained Willmore surface, then the Bäcklund transform Λ^* of Λ, of parameters α, L^α, is a real q^*-constrained Willmore surface, provided that it immerses.*

Note that, if Λ^* defines an immersion of M into $\mathbb{P}(\mathcal{L})$, then

$$\mathcal{C}_{\Lambda^*} = \mathcal{C} = \mathcal{C}_\Lambda.$$

The geometry of Bäcklund transformation is not clear.

Remark 6.35. *Note that*

$$p_{\alpha',\rho L'}(\lambda) = p_{\alpha',L'}(\lambda)^{-1}, \quad q_{\alpha',\rho L'}(\lambda) = q_{\alpha',L'}(\lambda)^{-1},$$

for all α', L' and $\lambda\in\mathbb{P}^1\backslash\{\pm\alpha'\}$. By (6.32), and having in consideration the reality of ρ, we get $q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(-\alpha) = q_{\overline{\alpha}^{-1},\rho\overline{L^\alpha}}(\alpha) = q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha)^{-1}$ and, therefore, by (6.33)

$$q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(-\alpha) = \rho\, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha)\,\rho, \tag{6.50}$$

making clear that the orthogonality of $\rho\, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha)L^\alpha$ *and* $q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha)L^\alpha$ *is equivalent to that of* $\rho\, q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(-\alpha)\rho L^\alpha$ *and* $q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(-\alpha)\rho L^\alpha$. *On the other hand, as we know,* L^α *is a* $d_V^{\alpha,q}$*-parallel null line bundle if and only if* ρL^α *is a* $d_V^{-\alpha,q}$*-parallel null line bundle. We conclude that* α, L^α *are Bäcklund transformation parameters if and only if so are* $-\alpha, \rho L^\alpha$. *Furthermore: the Bäcklund transforms of parameters* α, L^α *and* $-\alpha, \rho L^\alpha$ *coincide. In fact, according to equation* (6.50),

$$\tilde{L}^{-\alpha} = q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(-\alpha)\rho L^\alpha = \rho\, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\alpha)L^\alpha = \rho \tilde{L}^\alpha$$

and, therefore,

$$p_{-\alpha,\tilde{L}^{-\alpha}}(\lambda)\, q_{\overline{-\alpha}^{-1},\overline{\rho L^\alpha}}(\lambda) = p_{\alpha,\tilde{L}^\alpha_\beta}(\lambda)\, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\lambda),$$

for all $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha\}$.

We complete this section with an interesting and useful result relating the quadratic differentials q_Q and q_Q^*:

Proposition 6.36.

$$q_Q^* = q_Q.$$

PROOF. Fix z a holomorphic chart of $(M, \mathcal{C}_\Lambda)$. The proof will consist of showing that $(q^*)^z = q^z$, or, equivalently, that $q^z\tau^* = -2q^*_{\delta_z}((\mathcal{D}_{V^*})_{\delta_z}\tau^*)$, for all $\tau^* \in \Gamma((\Lambda^*)^{0,1})$. Fix $\tau \in \Gamma(\Lambda^{0,1})$ and set $\tau^* := r^*(1)^{-1}r^*(0)\tau$. Set also

$$\mathcal{R} := r^*(\infty)^{-1}r^*(0).$$

Let $\hat{d}$ be as defined in Section 6.5 for $r = r^*$. According to (6.4), followed by equation (6.19),

$$\begin{aligned} \mathcal{D}^{1,0}_{V^*} \circ r^*(1)^{-1}r^*(0) &= r^*(1)^{-1} \circ (\mathcal{D}^{\hat{d}}_V)^{1,0} \circ r^*(0) \\ &= r^*(1)^{-1}r^*(0)q^{1,0} + r^*(1)^{-1}r^*(\infty) \circ (\mathcal{D}^{1,0}_V - q^{1,0}) \circ \mathcal{R} \end{aligned}$$

and, therefore,

$$q^*_{\delta_z}((\mathcal{D}_{V^*})_{\delta_z}\tau^*) = r^*(1)^{-1}r^*(0)\, q_{\delta_z}\mathcal{R}^{-1} \circ ((\mathcal{D}_V)_{\delta_z} - q_{\delta_z}) \circ \mathcal{R}\,\tau,$$

as $q^{1,0}\Lambda^{0,1} = 0$. On the other hand,

$$\mathcal{R}^{-1} \circ ((\mathcal{D}_V)_{\delta_z} - q_{\delta_z}) \circ \mathcal{R} = \mathcal{R}^{-1} \circ ((\mathcal{D}_V)_{\delta_z} - q_{\delta_z})\mathcal{R} + (\mathcal{D}_V)_{\delta_z} - q_{\delta_z},$$

so we conclude that

$$q^*_{\delta_z}((\mathcal{D}_{V^*})_{\delta_z}\tau^*) = r^*(1)^{-1}r^*(0)\, q_{\delta_z}\mathcal{R}^{-1} \circ ((\mathcal{D}_V)_{\delta_z} - q_{\delta_z})\mathcal{R}\tau - \frac{1}{2}\, q^z\tau^*.$$

We complete this verification by showing that $\mathcal{R}^{-1} \circ ((\mathcal{D}_V)_{\delta_z} - q_{\delta_z})\mathcal{R}\,\Gamma(\Lambda^{0,1}) \subset \Gamma(\Lambda^{0,1})$, as follows. Set, more generally,

$$\mathcal{R}_\lambda := r^*(\infty)^{-1}r^*(\lambda).$$

In view of equation (6.20),

$$\mathcal{R}_\lambda^{-1} \circ (\mathcal{D}_V^{1,0} - q^{1,0})\mathcal{R}_\lambda = r^*(\lambda)^{-1} \circ ((\mathcal{D}_V^{\hat{d}})^{1,0} - \hat{q}^{1,0}) \circ r^*(\lambda) - \mathcal{D}_V^{1,0} + q^{1,0}.$$

On the other hand,

$$(\mathcal{D}_V^{\hat{d}})^{1,0} - \hat{q}^{1,0} = r^*(\lambda) \circ (\mathcal{D}_V^{1,0} - q^{1,0} + \lambda^{-1}\mathcal{N}_V^{1,0} + \lambda^{-2}q^{1,0}) \circ r^*(\lambda)^{-1} - \lambda^{-1}(\mathcal{N}_V^{\hat{d}})^{1,0} - \lambda^{-2}\hat{q}^{1,0}.$$

Thus

$$\mathcal{R}_\lambda^{-1} \circ (\mathcal{D}_V^{1,0} - q^{1,0})\mathcal{R}_\lambda = \mathrm{Ad}_{r^*(\lambda)^{-1}}(-\lambda^{-1}(\mathcal{N}_V^{\hat{d}})^{1,0} - \lambda^{-2}\hat{q}^{1,0} + \mathrm{Ad}_{r^*(\lambda)}(\lambda^{-1}\mathcal{N}^{1,0} + \lambda^{-2}q^{1,0})).$$

In view of the holomorphicity of $\mathrm{Ad}_{r^*(\lambda)}$ at $\lambda = 0$,

$$\mathrm{Ad}_{r^*(\lambda)} = \mathrm{Ad}_{r^*(0)} + \sum_{k \geq 1} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}$$

and, therefore, by (6.22),

$$\mathcal{R}_\lambda^{-1} \circ (\mathcal{D}_V^{1,0} - q^{1,0})\mathcal{R}_\lambda = \mathrm{Ad}_{r^*(\lambda)^{-1}}(\frac{d}{d\lambda}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}\mathcal{N}_V^{1,0} + \frac{1}{2}\frac{d^2}{d\lambda^2}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}q^{1,0} + o(\lambda)),$$

for λ near 0. Considering limits when λ goes to 0, we conclude that

$$\mathcal{R}^{-1} \circ (\mathcal{D}_V^{1,0} - q^{1,0})\mathcal{R} = \mathrm{Ad}_{r^*(0)^{-1}}(\frac{d}{d\lambda}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}\mathcal{N}_V^{1,0} + \frac{1}{2}\frac{d^2}{d\lambda^2}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}q^{1,0}).$$

For simplicity, set $\psi := r^*(0)^{-1}\frac{d}{d\lambda}_{|\lambda=0} r^*(\lambda) \in \Gamma(V \wedge V^\perp)$ (recalling (6.27)). Note that

$$\frac{d}{d\lambda}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}\mathcal{N}_V^{1,0} = \mathrm{Ad}_{r^*(0)}[\psi, \mathcal{N}_V^{1,0}].$$

The centrality of the central sphere congruence of Λ, $\mathcal{N}_V^{1,0}\Lambda^{0,1} = 0$, together with the skew-symmetry of $\mathcal{N}_V$, establishes that $\mathcal{N}_V^{1,0}$ takes values in the orthogonal to $\Lambda^{0,1}$. In particular,

$$\mathcal{N}_V^{1,0}V^\perp \subset V \cap (\Lambda^{0,1})^\perp = \Lambda^{0,1}. \tag{6.51}$$

It follows that

$$\mathrm{Ad}_{r^*(0)^{-1}}(\frac{d}{d\lambda}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}\mathcal{N}_V^{1,0})\Lambda^{0,1} = -\mathcal{N}_V^{1,0}\psi\Lambda^{0,1} \subset \Lambda^{0,1}.$$

On the other hand,

$$\mathrm{Ad}_{r^*(0)^{-1}}(\frac{d^2}{d\lambda^2}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}q^{1,0}) = r^*(0)^{-1}\frac{d^2}{d\lambda^2}_{|\lambda=0} r^*(\lambda)q^{1,0} - 2\psi q^{1,0}\psi + 2q^{1,0}\psi\psi$$

and, therefore,

$$\mathrm{Ad}_{r^*(0)^{-1}}(\frac{d^2}{d\lambda^2}_{|\lambda=0} \mathrm{Ad}_{r^*(\lambda)}q^{1,0})\Lambda^{0,1} = q^{1,0}\psi\psi\Lambda^{0,1},$$

as $q^{1,0}\Lambda^{0,1} = 0 = qV^\perp$. The fact that $q^{1,0}$ takes values in $\Lambda^{0,1}$ completes the proof. $\square$

6.7. Bäcklund transformation vs. spectral deformation

Bäcklund transformation and spectral deformation, of constrained harmonic bundles or complexified constrained Willmore surfaces, are closely related: as we verify in this section, the Bäcklund transform of parameters $\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \phi^\lambda L^\alpha, \phi^\lambda L^\beta$ of the spectral deformation of parameter λ, corresponding to a multiplier q, defined by ϕ^λ, coincides with the spectral deformation of parameter λ, corresponding to the multiplier q^*, of the Bäcklund transform of parameters $\alpha, \beta, L^\alpha, L^\beta$.

Suppose V is a q-constrained harmonic bundle, for some $q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)$. Let V^* be the Bäcklund transform of V of parameters $\alpha, \beta, L^\alpha, L^\beta$. Let r^* denote $r^{(\alpha,\beta)}_{L^\alpha, L^\beta}$ and, for each $\lambda \in \mathbb{C}\backslash\{0, \pm\alpha, \pm\beta\}$, let $\hat{d}^{\lambda,\hat{q}}_V$ be as defined in Section 6.5 for $r = r^*$. By definition of $\hat{d}^{\lambda,\hat{q}}_V$, $r^*(\lambda) : (\underline{\mathbb{C}}^{n+2}, d^{\lambda,q}_V) \to (\underline{\mathbb{C}}^{n+2}, \hat{d}^{\lambda,\hat{q}}_V)$ is an isometry of bundles preserving connections, for each λ. In particular, so is $r^*(1) : (\underline{\mathbb{C}}^{n+2}, d) \to (\underline{\mathbb{C}}^{n+2}, \hat{d})$. Fix λ in $\mathbb{C}\backslash\{0, \pm\alpha, \pm\beta\}$. In view of (6.5), we conclude that

$$d^{\lambda,q^*}_{V^*} = r^*(1)^{-1}\, r^*(\lambda) \circ d^{\lambda,q}_V \circ r^*(\lambda)^{-1} r^*(1).$$

It follows that, given

$$\phi^\lambda : (\underline{\mathbb{C}}^{n+2}, d^{\lambda,q}_V) \to (\underline{\mathbb{C}}^{n+2}, d),$$

an isometry preserving connections, so is

$$\psi^\lambda := \phi^\lambda\, r^*(\lambda)^{-1}\, r^*(1) : (\underline{\mathbb{C}}^{n+2}, d^{\lambda,q^*}_{V^*}) \to (\underline{\mathbb{C}}^{n+2}, d),$$

providing, therefore, the spectral deformation $\psi^\lambda V^*$, of parameter λ, corresponding to the multiplier q^*, of V^*. On the other hand, such isometry ϕ^λ provides the spectral deformation $\phi^\lambda V$, of parameter λ, corresponding to the multiplier q, of V. Next we focus on this $\mathrm{Ad}_{\phi^\lambda} q_\lambda$-constrained harmonic bundle. For simplicity, set $\tilde{q}_\lambda := \mathrm{Ad}_{\phi^\lambda} q_\lambda$. Let ρ_V and $\rho_{\phi^\lambda V}$ denote reflection across V and, respectively, $\phi^\lambda V$. Recalling (6.3), we get

$$d^{\frac{\alpha}{\lambda},\tilde{q}_\lambda}_{\phi^\lambda V} = \phi^\lambda \circ (d^{\lambda,q}_V)^{\frac{\alpha}{\lambda},q_\lambda}_V \circ (\phi^\lambda)^{-1} = \phi^\lambda \circ d^{\alpha,q}_V \circ (\phi^\lambda)^{-1},$$

which makes clear that, as L^α and L^β are, respectively, $d^{\alpha,q}_V$- and $d^{\beta,q}_V$-parallel, $\phi^\lambda L^\alpha$ and $\phi^\lambda L^\beta$ are, respectively, $d^{\frac{\alpha}{\lambda},\tilde{q}_\lambda}_{\phi^\lambda V}$- and $d^{\frac{\beta}{\lambda},\tilde{q}_\lambda}_{\phi^\lambda V}$-parallel. On the other hand, the fact that

$$\rho_{\phi^\lambda V} = \phi^\lambda \rho_V (\phi^\lambda)^{-1} \tag{6.52}$$

makes clear that, in view of the local non-orthogonality of L^β and $\rho_V L^\beta$, $\phi^\lambda L^\beta$ and $\rho_{\phi^\lambda V} \phi^\lambda L^\beta$ are, locally, non-orthogonal, as well. Equation (6.52) establishes, on the other hand,

$$p_{\phi^\lambda V, \alpha', \phi^\lambda L'}(\lambda) = \phi^\lambda\, p_{V,\alpha',L'}(\lambda)\, (\phi^\lambda)^{-1}, \quad q_{\phi^\lambda V, \alpha', \phi^\lambda L'}(\lambda) = \phi^\lambda\, q_{V,\alpha',L'}(\lambda)\, (\phi^\lambda)^{-1},$$

for all α', L' and $\lambda \neq \pm\alpha'$. Note that

$$p_{\alpha',L'}(\lambda) = p_{\frac{\alpha'}{\lambda},L'}(1), \quad q_{\alpha',L'}(\lambda) = q_{\frac{\alpha'}{\lambda},L'}(1).$$

for all α', L' and $\lambda \neq \pm\alpha'$. It follows that

$$\tilde{L}^{\alpha/\lambda}_{\beta/\lambda} = q_{\phi^\lambda V,\frac{\beta}{\lambda},\phi^\lambda L^\beta}(\frac{\alpha}{\lambda})\phi^\lambda L^\alpha = \phi^\lambda\, q_{V,\frac{\beta}{\lambda},L^\beta}(\frac{\alpha}{\lambda})L^\alpha = \phi^\lambda\, q_{V,\beta,L^\beta}(\alpha)L^\alpha = \phi^\lambda\,\tilde{L}^\alpha_\beta$$

establishing the local non-orthogonality of $\rho_{\phi^\lambda V}\tilde{L}^{\alpha/\lambda}_{\beta/\lambda} = \phi^\lambda\,\rho_V\,\tilde{L}^\alpha_\beta$ and $\tilde{L}^{\alpha/\lambda}_{\beta/\lambda}$, in view of that of $\rho_V\,\tilde{L}^\alpha_\beta$ and $\tilde{L}^\alpha_\beta$. We conclude that, for each λ, $\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \phi^\lambda L^\alpha, \phi^\lambda L^\beta$ constitute Bäcklund transformation parameters to $\phi^\lambda V$. Set

$$r^*_\lambda := p_{\phi^\lambda V,\frac{\alpha}{\lambda},\tilde{L}^{\alpha/\lambda}_{\beta/\lambda}}\, q_{\phi^\lambda V,\frac{\beta}{\lambda},\phi^\lambda L^\beta} = \phi^\lambda\, p_{V,\frac{\alpha}{\lambda},\tilde{L}^\alpha_\beta}\, q_{V,\frac{\beta}{\lambda},L^\beta}\,(\phi^\lambda)^{-1}.$$

Then

$$r^*_\lambda(1)^{-1}\,\phi^\lambda = \phi^\lambda\, q_{V,\beta,L^\beta}(\lambda)^{-1}\, p_{V,\alpha,\tilde{L}^\alpha_\beta}(\lambda)^{-1} = \phi^\lambda\, r^*(\lambda)^{-1}$$

and, therefore, $r^*_\lambda(1)^{-1}\,\phi^\lambda = \psi^\lambda\, r^*(1)^{-1}$. In particular,

$$r^*_\lambda(1)^{-1}\,\phi^\lambda V = \psi^\lambda\, r^*(1)^{-1}V,$$

the Bäcklund transform of parameters $\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \phi^\lambda L^\alpha, \phi^\lambda L^\beta$ of the spectral deformation $\phi^\lambda V$ of V, of parameter λ [corresponding to the multiplier q], coincides with the spectral deformation of parameter λ [corresponding to the multiplier q^*] of the Bäcklund transform of V of parameters $\alpha, \beta, L^\alpha, L^\beta$.

This permutability between Bäcklund transformation and spectral deformation of constrained harmonic bundles extends to constrained Willmore surfaces. Indeed, if V is a (q,d)-central sphere congruence to some q-constrained Willmore surface $(\Delta^{1,0}, \Delta^{0,1})$, the equality

$$r^*_\lambda(1)^{-1} = \phi^\lambda\, r^*(\lambda)^{-1}\,(\phi^\lambda)^{-1}$$

establishes, furthermore,

$$\begin{aligned} r^*_\lambda(1)^{-1} r^*_\lambda(\infty)\,\phi^\lambda\Delta^{1,0} &= \phi^\lambda r^*(\lambda)^{-1} p_{V,\frac{\alpha}{\lambda},\tilde{L}^\alpha_\beta}(\infty)\Delta^{1,0} \\ &= \phi^\lambda r^*(\lambda)^{-1} p_{V,\alpha,\tilde{L}^\alpha_\beta}(\infty)\Delta^{1,0} \end{aligned}$$

and, therefore,

$$r^*_\lambda(1)^{-1} r^*_\lambda(\infty)\,\phi^\lambda\Delta^{1,0} = \psi^\lambda\, r^*(1)^{-1} r^*(\infty)\Delta^{1,0};$$

and, similarly,

$$r^*_\lambda(1)^{-1} r^*_\lambda(0)\,\phi^\lambda\Delta^{0,1} = \psi^\lambda\, r^*(1)^{-1} r^*(0)\Delta^{0,1}.$$

Theorem 6.37. *Let V be a q-constrained harmonic bundle, α,β,L^α,L^β be Bäcklund transformation parameters to V, $\lambda \in \mathbb{C}\backslash\{0, \pm\alpha, \pm\beta\}$ and $\phi^\lambda : (\underline{\mathbb{C}}^{n+2}, d^{\lambda,q}_V) \to (\underline{\mathbb{C}}^{n+2}, d)$ be an isometry preserving connections. The Bäcklund transform of parameters $\frac{\alpha}{\lambda}$, $\frac{\beta}{\lambda}$, $\phi^\lambda L^\alpha$, $\phi^\lambda L^\beta$ of the spectral deformation $\phi^\lambda V$ of V, of parameter λ [corresponding to*

the multiplier q], coincides with the spectral deformation of parameter λ [corresponding to the multiplier q^*] of the Bäcklund transform of parameters $\alpha, \beta, L^\alpha, L^\beta$ of V. Furthermore, if V is a q-central sphere congruence to a constrained Willmore surface $(\Delta^{1,0}, \Delta^{0,1})$ and $\phi_*^\lambda : (\underline{\mathbb{C}}^{n+2}, d_{V^*}^{\lambda, q^*}) \to (\underline{\mathbb{C}}^{n+2}, d)$ is an isometry preserving connections, then the diagram in Figure 6-2 commutes.

$$\begin{tikzcd}
 & (\phi_*^\lambda(\Delta^*)^{1,0}, \phi_*^\lambda(\Delta^*)^{0,1}) = (\phi^\lambda \Delta^{1,0}, \phi^\lambda \Delta^{0,1})^* & \\
(\Delta^{1,0}, \Delta^{0,1})^* \arrow[ur, dash, "\lambda"] \arrow[dr, dash, "{\alpha, \beta, L^\alpha, L^\beta}"'] & & (\phi^\lambda \Delta^{1,0}, \phi^\lambda \Delta^{0,1}) \arrow[ul, dash, "{\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \phi^\lambda L^\alpha, \phi^\lambda L^\beta}"'] \\
 & (\Delta^{1,0}, \Delta^{0,1}) \arrow[ur, dash, "\lambda"'] &
\end{tikzcd}$$

FIGURE 6-2. A Bianchi permutability of spectral deformation and Bäcklund transformation of constrained Willmore surfaces.

For $\lambda \in \{\pm\alpha, \pm\beta\}$, it is not clear how the spectral deformation of parameter λ relates to the Bäcklund transformation of parameters $\alpha, \beta, L^\alpha, L^\beta$.

CHAPTER 7

Constrained Willmore surfaces with a conserved quantity

A powerful result by E. Noether [**47**] establishes that any symmetry of the action of a physical system has a corresponding conservation law. Time translation symmetry gives conservation of energy, space translation symmetry gives conservation of momentum, symmetry under rotation gives conservation of angular momentum - these are examples of physically conserved quantities one gets from symmetries of the laws of nature. Noether's theorem has become a fundamental tool of modern theoretical physics and the calculus of variations. In this chapter, we introduce the concept of *conserved quantity* of a constrained Willmore surface in a space-form, an idea by Fran Burstall and David Calderbank. Constrained Willmore surfaces in space-forms admitting a conserved quantity form a subclass of constrained Willmore surfaces preserved by both spectral deformation and Bäcklund transformation, for special choices of parameters. In codimension 1, this class consists of the class of constant mean curvature surfaces in space-forms. In codimension 2, surfaces with holomorphic mean curvature vector in some space-form are examples of constrained Willmore surfaces admitting a conserved quantity.

Let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a q-constrained Willmore surface in the projectivized light-cone, for some real 1-form q with values in $\Lambda \wedge \Lambda^{(1)}$. Let S be the central sphere congruence of Λ. Provide M with the conformal structure $\mathcal{C}_\Lambda$, induced by Λ.

7.1. Conserved quantities of constrained Willmore surfaces

Let
$$p(\lambda) := \lambda^{-1}v + v_0 + \lambda\overline{v}$$
be a Laurent polynomial with $v_0 \in \Gamma(S)$ real, $v \in \Gamma(S^\perp)$ and $v_\infty := p(1) = v_0 + v + \overline{v} \neq 0$.

Definition 7.1. *We say that $p(\lambda)$ is a q-*conserved quantity *of Λ if*
$$d_S^{\lambda,q}p(\lambda) = 0, \tag{7.1}$$
for all $\lambda \in \mathbb{C}\backslash\{0\}$.

Theorem 7.2. *$p(\lambda)$ is a q-conserved quantity of Λ if and only if*
$$dv_\infty = 0, \quad \mathcal{D}^{0,1}v = 0, \quad \mathcal{N}^{1,0}v + q^{1,0}v_0 = 0.$$

PROOF. Given $\lambda \in \mathbb{C}\backslash\{0\}$,

$$\begin{aligned} d_S^{\lambda,q}p(\lambda) &= \lambda^{-1}\mathcal{D}v + \mathcal{D}v_0 + \lambda\mathcal{D}\overline{v} + \lambda^{-2}\mathcal{N}^{1,0}v + \lambda^{-1}\mathcal{N}^{1,0}v_0 + \mathcal{N}^{1,0}\overline{v} \\ &\quad + \mathcal{N}^{0,1}v + \lambda\mathcal{N}^{0,1}v_0 + \lambda^2\mathcal{N}^{0,1}\overline{v} + (\lambda^{-2}-1)q^{1,0}v_0 + (\lambda^2-1)q^{0,1}v_0. \end{aligned}$$

Organizing the terms in $d_S^{\lambda,q}p(\lambda)$ by powers of λ shows that equation (7.1) holds for all $\lambda \in \mathbb{C}\backslash\{0\}$ if and only if so do equations $\mathcal{D}v + \mathcal{N}^{1,0}v_0 = 0 = \mathcal{D}\overline{v} + \mathcal{N}^{0,1}v_0$, together with $\mathcal{N}^{1,0}v + q^{1,0}v_0 = 0 = \mathcal{N}^{0,1}\overline{v} + q^{0,1}v_0$ and with

$$\mathcal{D}v_0 + \mathcal{N}^{1,0}\overline{v} + \mathcal{N}^{0,1}v - q^{1,0}v_0 - q^{0,1}v_0 = 0. \tag{7.2}$$

In view of the reality of $\mathcal{D}$, $\mathcal{N}$ and q, we conclude that equation (7.1) holds if and only if so do equations (7.2),

$$\mathcal{D}v + \mathcal{N}^{1,0}v_0 = 0 \tag{7.3}$$

and

$$\mathcal{N}^{1,0}v + q^{1,0}v_0 = 0. \tag{7.4}$$

Equation (7.3) is equivalent to the system of equations

$$\mathcal{D}^{1,0}v + \mathcal{N}^{1,0}v_0 = 0 \tag{7.5}$$

and

$$\mathcal{D}^{0,1}v = 0. \tag{7.6}$$

In the light of (7.4), equation (7.2) reads $\mathcal{D}v_0 + \mathcal{N}^{1,0}\overline{v} + \mathcal{N}^{0,1}v + \mathcal{N}^{1,0}v + \mathcal{N}^{0,1}\overline{v} = 0$, i.e.,

$$\mathcal{D}v_0 + \mathcal{N}(v + \overline{v}) = 0. \tag{7.7}$$

On the other hand, in view of equations (7.5), (7.6) and (7.7),

$$\begin{aligned} dv_\infty &= \mathcal{D}v_0 + \mathcal{D}(v+\overline{v}) + \mathcal{N}v_0 + \mathcal{N}(v+\overline{v}) \\ &= \mathcal{D}^{1,0}v + \mathcal{D}^{0,1}v + \mathcal{D}^{1,0}\overline{v} + \mathcal{D}^{0,1}\overline{v} + \mathcal{N}^{1,0}v_0 + \mathcal{N}^{0,1}v_0 \\ &= \overline{\mathcal{D}^{0,1}v} + \overline{\mathcal{D}^{1,0}v + \mathcal{N}^{1,0}v_0} \end{aligned}$$

and, ultimately,

$$dv_\infty = 0. \tag{7.8}$$

We complete the proof by observing that, together with equation (7.6), equation (7.8) establishes (7.5) and (7.7). For that, first note that $\pi_S(dv_\infty) = \mathcal{D}v_0 + \mathcal{N}(v+\overline{v})$ and $\pi_{S^\perp}(dv_\infty) = \mathcal{D}(v+\overline{v}) + \mathcal{N}v_0$. Thus dv_∞ vanishes if and only if

$$\mathcal{D}v_0 + \mathcal{N}(v+\overline{v}) = 0 = \mathcal{D}(v+\overline{v}) + \mathcal{N}v_0.$$

In particular, $dv_\infty = 0$ forces $0 = \mathcal{N}^{1,0}v_0 + \mathcal{D}^{1,0}v + \mathcal{D}^{1,0}\overline{v}$ which, together with equation (7.6), establishes equation (7.5), completing the proof. $\square$

For later reference,

Remark 7.3. *The characterization of the $d_S^{\lambda,q}$-parallelism of $p(\lambda)$ by the equations in Theorem 7.2 does not involve the fact that Λ is q-constrained Willmore, only the reality of $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$.*

Remark 7.4. *In the characterization above, of a q-conserved quantity of Λ, equation $\mathcal{N}^{1,0}v + q^{1,0}v_0 = 0$ determines q. In fact, q is real, so it is determined by $q^{1,0}$, which, given that Λ is a q-constrained Willmore surface, is ensured to be a 1-form with values in $\Lambda \wedge \Lambda^{0,1}$. Hence $q^{1,0}$ is determined by $q^{1,0}u \in \Gamma(\Lambda^{0,1})$, fixing $u \in S\backslash\Lambda^\perp$ (cf. Remark 5.8). On the other hand, $\mathcal{N}^{1,0}v \in \Gamma(\Lambda^{0,1})$ (cf. (6.51)). Finally, if $dv_\infty = 0$, then v_∞ defines a space-form S_{v_∞}, and, for σ_∞ the surface in S_{v_∞} defined by Λ, $(\sigma_\infty, v_0) = (\sigma_\infty, \pi_S(v_\infty)) = (\sigma_\infty, v_\infty) = -1$ is never-zero.*

Remark 7.5. *If $p(\lambda)$ is a q-conserved quantity of Λ, then*

$$d(v, v) = 0.$$

Indeed, by equation (7.4), and in view of the skew-symmetry of q,

$$(\mathcal{N}^{1,0}v, v_0) = -(q^{1,0}v_0, v_0) = (v_0, q^{1,0}v_0) = -(v_0, \mathcal{N}^{1,0}v)$$

and, therefore, $(\mathcal{N}^{1,0}v, v_0) = 0$. Equivalently, $(v, \mathcal{N}^{1,0}v_0) = 0$, in view of the skew-symmetry of $\mathcal{N}$, which, together with equation (7.5), establishes $(v, \mathcal{D}^{1,0}v) = 0$ and, consequently, by equation (7.6), $(v, \mathcal{D}v) = 0$.

7.2. Examples

Two special cases follow from the characterization of a conserved quantity provided by Theorem 7.2.

7.2.1. The special case of codimension 1: CMC surfaces in 3-space. The existence of a conserved quantity $p(\lambda)$ of Λ establishes, in particular, the constancy of $v_\infty := p(1)$. In the particular case of $n = 3$, we verify that Λ has constant mean curvature in the space-form S_{v_∞}, that is, the surface defined by Λ in the space-form S_{v_∞} has constant mean curvature. In fact, constant mean curvature surfaces in 3-dimensional space-forms are, precisely, the constrained Willmore surfaces in 3-space-form admitting a conserved quantity. This case will be addressed in detail in Section 8.2, dedicated to constant mean curvature surfaces in 3-dimensional space-forms.

7.2.2. A special case in codimension 2: holomorphic mean curvature vector surfaces in 4-space. In codimension 2, the complexification of $S^\perp$ admits a unique decomposition into the direct sum of two null complex lines, complex conjugate of each other: given $v\in\Gamma(S^\perp)$ null, $S^\perp=\langle v\rangle\oplus\langle\overline{v}\rangle$. Such a v defines an almost-complex structure J_v on $S^\perp$, with eigenvalues i and $-i$ and eigenspaces $\langle v\rangle$ and $\langle\overline{v}\rangle$, associated to i and $-i$, respectively. In [**11**] (see, in particular, Corollary 14.3), F. Burstall and D. Calderbank proved that a codimension 2 surface in a space-form, with holomorphic mean curvature vector with respect to the complex structure induced by $\nabla^{S^\perp}$, is constrained Willmore. In this section, we prove it in our setting, proving, furthermore, that, in 4-dimensional space-form, the constrained Willmore surfaces admitting a conserved quantity $p(\lambda)=\lambda^{-1}v+v_0+\lambda\overline{v}$ with v null are the surfaces with holomorphic mean curvature vector in the space-form $S_{p(1)}$, with respect to the complex structure on $(S^\perp,J_v,\nabla^{S^\perp})$ determined by Koszul-Malgrange Theorem.

Suppose $n=4$. In that case, $S^\perp$ is a (non-degenerate real) rank 2 bundle, admitting, therefore, a unique decomposition

$$S^\perp=S^\perp_+\oplus S^\perp_- \tag{7.9}$$

of its complexification into the direct sum of two null complex lines, complex conjugate of each other. In particular, $S^\perp$ admits an almost-complex structure (in fact, two, differing by sign),

$$J_{S^\perp}=\pm I\begin{cases} i & \text{on } S^\perp_+\\ -i & \text{on } S^\perp_-\end{cases}.$$

Provide then $S^\perp$ with the unique complex structure compatible with the connection $\nabla^{S^\perp}=\mathcal{D}|_{\Gamma(S^\perp)}$, cf. Koszul-Malgrange Theorem, characterized by the fact that a section ν of $S^\perp$ is holomorphic if and only if $\mathcal{D}^{0,1}\nu=0$ (see, for example, [**15**], Theorem 2.1). Fix a choice of $J_{S^\perp}$ and provide $S^\perp$ with the structure of complex vector bundle defined by $i\nu:=J_{S^\perp}\nu$, for all real $\nu\in\Gamma(S^\perp)$. Then, given $\nu\in\Gamma(S^\perp)$ real and $z=x+iy$ a holomorphic chart of M,

$$\mathcal{D}_{\delta_{\bar z}}\nu=\frac{1}{2}\,(\mathcal{D}_{\delta_x}\nu+J_{S^\perp}\mathcal{D}_{\delta_y}\nu). \tag{7.10}$$

Writing $\nu=v+\overline{v}$, with v in the eigenspace of $J_{S^\perp}$ associated to the eigenvalue i, and expanding (7.10) out, we conclude that ν is holomorphic if and only if $\mathcal{D}^{0,1}v+\mathcal{D}^{1,0}\overline{v}=0$, or, equivalently, $\mathcal{D}^{0,1}v=0$.

Now fix a non-zero $v_\infty\in\mathbb{R}^{5,1}$. Let $v_\infty^\perp$ be the orthogonal projection of v_∞ onto $S^\perp$. In view of the reality of $v_\infty^\perp$, write $v_\infty^\perp=v+\overline{v}$, with v in the eigenspace of $J_{S^\perp}$ associated to the eigenvalue i. Consider the surface $\sigma_\infty:M\to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. Under the isomorphism $\mathcal{Q}:N_\infty\to S^\perp$, preserving connections, defined in Section 2.2, the complex structure on $S^\perp$ induces naturally a complex structure on

N_∞, preserving holomorphicity. We say that Λ has *holomorphic mean curvature vector* in the space-form S_{v_∞} if $\mathcal{H}_\infty$ is holomorphic. By equation (4.2), it follows that Λ is a holomorphic mean curvature vector surface in S_{v_∞} if and only if

$$\mathcal{D}^{0,1}v_\infty^\perp = 0, \tag{7.11}$$

or, equivalently, $\mathcal{D}^{0,1}v = 0$.

Now suppose Λ has holomorphic mean curvature vector in S_{v_∞}. Define a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, with $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$, by setting $q^{1,0}v_\infty^T := -\mathcal{N}^{1,0}v$, for v_∞^T the orthogonal projection of v_∞ onto S (cf. Remark 7.4). Set $p(\lambda) := \lambda^{-1}v + v_\infty^T + \lambda\overline{v}$. According to Theorem 7.2, having in consideration Remark 7.3, $d_S^{\lambda,q}p(\lambda) = 0$ and, consequently,

$$R^\lambda p(\lambda) = 0, \tag{7.12}$$

for R^λ the curvature tensor of $d_S^{\lambda,q}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$. In the proof of Theorem 6.6, we observed, in particular, that

$$R^\lambda = \frac{\lambda^{-1}-\lambda}{2}\, i\,(d^{\mathcal{D}} * \mathcal{N} - 2[q \wedge *\mathcal{N}]) + (\lambda^{-2}-1)\, d^{\mathcal{D}}q^{1,0} + (\lambda^2-1)\, d^{\mathcal{D}}q^{0,1},$$

having in consideration that, as q is real and $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$, $[q \wedge q]$ vanishes (cf. (**??**)). Note that, as $qS^\perp = 0$ and $S^\perp$ is $\mathcal{D}$-parallel, $d^{\mathcal{D}}q^{1,0}S^\perp = 0 = d^{\mathcal{D}}q^{0,1}S^\perp$. According to (6.1) and (6.2), equation (7.12) establishes

$$\frac{\lambda^{-1}-\lambda}{2}\,(d^{\mathcal{D}} * \mathcal{N} - 2[q \wedge *\mathcal{N}])v_\infty^T = 0 \tag{7.13}$$

and

$$0 = \frac{i}{2}(d^{\mathcal{D}} * \mathcal{N} - 2[q \wedge *\mathcal{N}])((\lambda^{-2}-1)v + (1-\lambda^2)\overline{v}) + (\lambda^{-2}-1)d^{\mathcal{D}}q^{1,0}v_\infty^T + (\lambda^2-1)d^{\mathcal{D}}q^{0,1}v_\infty^T.$$

Organizing the terms in equation (7.13) by powers of λ, we conclude from the fact that equation (7.12) holds for all $\lambda \in \mathbb{C}\backslash\{0\}$ that, in particular,

$$d^{\mathcal{D}} * \mathcal{N} - 2[q \wedge *\mathcal{N}]v_\infty^T = 0, \tag{7.14}$$

or, equivalently, cf. Remark 5.7, $d^{\mathcal{D}} * \mathcal{N} = 2[q \wedge *\mathcal{N}]$, in view of the fact that $(\sigma_\infty, v_\infty^T) = (\sigma_\infty, v_\infty)(= -1)$ is never-zero. Now we see that equation (7.12) establishes, furthermore,

$$(\lambda^{-2}-1)\, d^{\mathcal{D}}q^{1,0}v_\infty^T + (\lambda^2-1)\, d^{\mathcal{D}}q^{0,1}v_\infty^T = 0$$

and the fact that it does not depend on $\lambda \in \mathbb{C}\backslash\{0\}$ establishes then $d^{\mathcal{D}}q^{1,0}v_\infty^T = 0$. Lastly, observe, in view of (5.8), that, as $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$, then so does $d^{\mathcal{D}}q^{1,0}$,

$$d^{\mathcal{D}}q^{1,0} \in \Omega^2(\Lambda \wedge \Lambda^{0,1}),$$

to conclude, cf. Remark 5.8, that $d^{\mathcal{D}}q^{1,0} = 0$ and, ultimately, by Lemma 5.10, that $d^{\mathcal{D}}q = 0$. We conclude that Λ is a q-constrained Willmore surface admitting $p(\lambda)$ as a q-conserved quantity.

Conversely, suppose Λ is a constrained Willmore surface admitting a conserved quantity $p(\lambda) = \lambda^{-1}w + v_0 + \lambda\overline{w}$ with $v_0 \in \Gamma(S)$ real and $w \in \Gamma(S^\perp)$ null (and, in particular, never-zero). In that case, $S^\perp = \langle w\rangle \oplus \langle\overline{w}\rangle$ is the decomposition of $S^\perp$ in (7.9). Let J_w be the almost-complex structure on $S^\perp$ admitting $\langle w\rangle$ and $\langle\overline{w}\rangle$ as eigenspaces associated to the eigenvalues i and $-i$, respectively. According to the characterization of conserved quantities presented in Theorem 7.2, we conclude that Λ has holomorphic mean curvature vector in $S_{p(1)}$, with respect to the complex structure on $(S^\perp, J_w, \nabla^{S^\perp})$ determined by Koszul-Malgrange Theorem.

7.3. Spectral deformation of constrained Willmore surfaces with a conserved quantity

The spectral deformation of constrained Willmore surfaces preserves the existence of a conserved quantity, trivially:

Theorem 7.6. *Let μ be in S^1 and $\phi^\mu_q : (\underline{\mathbb{R}}^{n+1,1}, d^{\mu,q}_S) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ be an isomorphism. Suppose that either v_0 is non-zero or $\overline{\mu}v + \mu\overline{v}$ is non-zero. In that case, if $p(\lambda)$ is a q-conserved quantity of Λ, then $\phi^\mu_q p(\mu\lambda)$ is a $\mathrm{Ad}_{\phi^\mu_q}(q_\mu)$-conserved quantity of the spectral deformation $\phi^\mu_q\Lambda$ of parameter μ of Λ.*

Proof. By hypothesis,

$$v^\mu_\infty := \phi^\mu_q(\overline{\mu}v + v_0 + \mu\overline{v})$$

is non-zero. On the other hand, as ϕ^μ_q is real and μ is unit, we have $\overline{\mu^{-1}\phi^\mu_q v} = \mu\,\phi^\mu_q\overline{v}$. Having in consideration that ϕ^μ_q is an isometry, and, in particular, $(\phi^\mu_q S)^\perp = \phi^\mu_q S^\perp$, we conclude that

$$\phi^\mu_q p(\mu\lambda) = \lambda^{-1}(\mu^{-1}\phi^\mu_q v) + \phi^\mu_q v_0 + \lambda(\mu\,\phi^\mu_q\overline{v})$$

is of the right form. The fact that $\phi^\mu_q : (\underline{\mathbb{R}}^{n+1,1}, d^{\mu,q}_S) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ preserves connections, and, consequently,

$$d^{\lambda,\mathrm{Ad}_{\phi^\mu_q}(q_\mu)}_{\phi^\mu_q S} = \phi^\mu_q \circ (d^{\mu,q}_S)^{\lambda,q_\mu}_S \circ (\phi^\mu_q)^{-1} = \phi^\mu_q \circ d^{\mu\lambda,q}_S \circ (\phi^\mu_q)^{-1},$$

completes the proof. □

7.4. Bäcklund transformation of constrained Willmore surfaces with a conserved quantity

Bäcklund transformations of constrained Willmore surfaces preserve the existence of a conserved quantity, in the following terms:

Theorem 7.7. *Suppose $p(\lambda)$ is a q-conserved quantity of Λ. Let α, L^α be Bäcklund transformation parameters to Λ corresponding to the multiplier q and let r^* denote $r^\alpha_{L^\alpha}$. If*

$$p(\alpha) \perp L^\alpha, \tag{7.15}$$

then

$$p^*(\lambda) := r^*(1)^{-1}\, r^*(\lambda)\, p(\lambda)$$

is a q^*-conserved quantity of the Bäcklund transform Λ^* of Λ of parameters α, L^α, provided that Λ^* immerses.

Proof. Suppose Λ^* immerses and let S^* be its central sphere congruence. First of all, note that

$$d^{\lambda,q^*}_{S^*}\, p^*(\lambda) = r^*(1)^{-1} \circ \hat{d}^{\lambda,\hat{q}}_{S} \circ r^*(\lambda)\, p(\lambda) = r^*(1)^{-1} r^*(\lambda) \circ d^{\lambda,q}_{S}\, p(\lambda) = 0,$$

for $\lambda \in \mathbb{C}\backslash\{0\}$. Let ρ and ρ^* denote, respectively, ρ_S and ρ_{S^*}. Consider projections $\pi_{L^\alpha} : \underline{\mathbb{C}}^{n+2} \to L^\alpha$, $\pi_{\rho L^\alpha} : \underline{\mathbb{C}}^{n+2} \to \rho L^\alpha$ and $\pi_\perp : \underline{\mathbb{C}}^{n+2} \to (L^\alpha \oplus \rho L^\alpha)^\perp$ with respect to the decomposition $\underline{\mathbb{C}}^{n+2} = L^\alpha \oplus \rho L^\alpha \oplus (L^\alpha \oplus \rho L^\alpha)^\perp$. As L^α and ρL^α are never orthogonal, condition (7.15) establishes, in particular, $\pi_{\rho L^\alpha} p(\alpha) = 0$. On the other hand, in view of the specific form of $p(\lambda)$,

$$\rho p(\lambda) = p(-\lambda), \tag{7.16}$$

for all λ. Hence $\pi_{L^\alpha} p(-\alpha) = \rho \pi_{\rho L^\alpha} p(\alpha)$ and, therefore, $\pi_{L^\alpha} p(-\alpha) = 0$. It follows that

$$p_{\alpha,L^\alpha}(\lambda)\, p(\lambda) = \frac{\alpha-\lambda}{\alpha+\lambda}\, \pi_{L^\alpha} p(\lambda) + \pi_\perp p(\lambda) + \frac{\alpha+\lambda}{\alpha-\lambda}\, \pi_{\rho L^\alpha} p(\lambda)$$

has no poles and, consequently, that

$$p^*(\lambda) = r^*(1)^{-1} K q_{\overline{\alpha}^{-1},\hat{L}^{\overline{\alpha}^{-1}}}(\lambda)\, p_{\alpha,L_\alpha}(\lambda)\, p(\lambda)$$

has, at most, poles at $\lambda = \pm\overline{\alpha}^{-1}$. Consider now projections $\pi_{\overline{L^\alpha}} : \underline{\mathbb{C}}^{n+2} \to \overline{L^\alpha}$ and $\pi_{\rho\overline{L^\alpha}} : \underline{\mathbb{C}}^{n+2} \to \rho\overline{L^\alpha}$ with respect to the decomposition $\underline{\mathbb{C}}^{n+2} = \overline{L^\alpha} \oplus \rho\overline{L^\alpha} \oplus (\overline{L^\alpha} \oplus \rho\overline{L^\alpha})^\perp$. Given the specific form of $p(\lambda)$, we have, on the other hand,

$$p(\overline{\lambda}^{-1}) = \overline{p(\lambda)}, \tag{7.17}$$

for all λ, and, therefore, $p(\overline{\alpha}^{-1}) \in \Gamma(\overline{L^\alpha}^{\perp})$. Thus $\pi_{\overline{L^\alpha}}\, p(-\overline{\alpha}^{-1}) = 0 = \pi_{\rho\overline{L^\alpha}}\, p(\overline{\alpha}^{-1})$. It follows that $p^*(\lambda) = r^*(1)^{-1}\, p_{\alpha,\tilde{L}^\alpha}(\lambda)\, q_{\overline{\alpha}^{-1},\overline{L^\alpha}}(\lambda)\, p(\lambda)$ has, at most, poles at $\lambda = \pm\alpha$. We conclude that $p^*(\lambda)$ has no poles. The fact that $\lim_{\lambda\to\infty} \lambda^{-1} p^*(\lambda) = r^*(1)^{-1}\, r^*(\infty)\, \overline{v}$ and $\lim_{\lambda\to 0} \lambda p^*(\lambda) = r^*(1)^{-1}\, r^*(0)\, v$ are both finite establishes then $p^*(\lambda)$ as a degree 1 Laurent polynomial. According to (6.16),

$$\rho^* p^*(\lambda) = r^*(1)^{-1} \rho\, r^*(1) p^*(\lambda) = r^*(1)^{-1} \rho\, r^*(\lambda)\, p(\lambda) = r^*(1)^{-1} r^*(-\lambda) \rho p(\lambda)$$

and, therefore, following (7.16),

$$\rho^* p^*(\lambda) = p^*(-\lambda),$$

showing that the coefficients on λ and λ^{-1} in $p^*(\lambda)$ are sections of $(S^*)^\perp$, whilst the coefficient on λ^0 is a section of S^*. To complete the proof, we are left to verify that $p^*(\overline{\lambda}^{-1}) = \overline{p^*(\lambda)}$, equivalent to the complex conjugation conditions on the coefficients in $p^*(\lambda)$. It comes as an immediate consequence of equations (6.48) and (7.17). $\square$

CHAPTER 8

Constrained Willmore surfaces and isothermic condition

A classical result of Thomsen [**55**] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. Constant mean curvature (CMC) surfaces in 3-dimensional space-forms are, in particular, isothermic constrained Willmore surfaces, as proven by J. Richter [**51**]. However, isothermic constrained Willmore surfaces in 3-space are not necessarily CMC surfaces in some space-form, as proven by an example due to Fran Burstall and presented in [**7**], of a constrained Willmore cylinder that does not have constant mean curvature in any space-form. We dedicate a section to the very important class of CMC surfaces in 3-space, with constrained Willmore Bäcklund transformations; both constrained Willmore and isothermic spectral deformations; as well as a spectral deformation of their own and, in the Euclidean case, isothermic Darboux transformations and Bianchi-Bäcklund transformations. S. Kobayashi and J.-I. Inoguchi [**35**] proved that isothermic Darboux transformation of a CMC surface in $\mathbb{R}^3$ is equivalent to Bianchi-Bäcklund transformation. We believe isothermic Darboux transformation of a CMC surface in Euclidean 3-space can be obtained as a particular case of constrained Willmore Bäcklund transformation. This shall be the subject of further work. We present the classical CMC spectral deformation by means of the action of a loop of flat metric connections. We observe that these three spectral deformations of CMC surfaces in 3-space are all closely related and, therefore, all closely related to Bäcklund transformation. We observe, in particular, that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation and that, in the particular case of minimal surfaces, the classical CMC spectral deformation coincides with the constrained Willmore spectral deformation corresponding to the zero multiplier. The chapter starts with a section on the Möbius invariant class of isothermic surfaces in space forms. We characterize isothermic constrained Willmore surfaces by the non-uniqueness of multiplier. The constrained Willmore spectral deformation is known to preserve the isothermic condition, cf. [**14**]. As for Bäcklund transformation of constrained Willmore surfaces, we believe it does not necessarily preserve the isothermic condition. We believe one can obtain non-isothermic, non-Willmore constrained Willmore surfaces as Bäcklund transforms of non-minimal CMC surfaces in space-forms. This shall be the subject of further work.

Throughout this chapter, let $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ be a surface in the projectivized light-cone and S be the central sphere congruence of Λ. Consider M provided with the conformal structure $\mathcal{C}_\Lambda$.

8.1. Isothermic surfaces

It seems that the notion of isothermal lines, tracing back to the early nineteenth century, was motivated by their physical interpretation as lines of equal temperature, having led to the notion of isothermic surfaces, that is, surfaces with isothermal lines of curvature. This section is dedicated to the study of isothermic surfaces merely from the point of view of constrained Willmore surfaces. Classically, a surface in $\mathbb{R}^3$ is isothermic if it admits conformal coordinate line coordinates at every point. F. Burstall and U. Pinkall [**13**] extended the isothermic condition to surfaces in space forms, with a manifestly conformally formulation, characterizing isothermic surfaces in the conformal n-sphere by the existence of a non-zero real closed 1-form η with values in a certain subbundle of the skew-symmetric endomorphisms of $\underline{\mathbb{R}}^{n+1,1}$. We characterize isothermic constrained Willmore surfaces by the non-uniqueness of multiplier and establish the set of multipliers to an isothermic q-constrained Willmore surface (Λ, η) as the 1-dimensional affine space $q + \langle *\eta \rangle_{\mathbb{R}}$. The constrained Willmore spectral deformation is known to preserve the isothermic condition, cf. [**14**]. We derive it in our setting. As for Bäcklund transformation of constrained Willmore surfaces, it is not clear that the isothermic condition is preserved. Isothermic surfaces in $\mathbb{R}^3$ were studied intensively at the turn of the 20th century and a rich transformation theory of these surfaces was developed in the works of Darboux [**22**], Calapso [**16**], [**17**] and Bianchi [**2**], [**3**]. The loop group formalism provides a context in which the results of Bianchi, Calapso and Darboux can be generalized. Following the work of F. Burstall, D. Calderbank and U. Pinkall [**11**], [**13**], we characterize isothermic surfaces by the flatness of a certain $\mathbb{R}$-family of metric connections on $\underline{\mathbb{R}}^{n+1,1}$ and define, in terms of this family of connections, both the isothermic spectral deformation, discovered in the classical setting by Calapso and, independently, by Bianchi; and the isothermic Darboux transformation.

8.1.1. Isothermic surfaces: definition. In this section, we present a manifestly conformally invariant formulation of the isothermic condition, by F. Burstall and U. Pinkall.

Isothermic surfaces are classically defined to be immersions $f : (M, g_f) \to \mathbb{R}^3$ admitting, at every point, conformal curvature line coordinates, i.e., conformal coordinates along the principal directions. Equation (2.2), relating the shape operator to the second fundamental form of an isometric immersion, makes clear that conformal coordinates

x and y are curvature line coordinates if and only if

$$\Pi(\delta_x, \delta_y) = 0. \tag{8.1}$$

In fact, the conformality of x and y ensures the existence of some $u \in C^\infty(M, \mathbb{R})$ for which

$$g_f = e^u(dx^2 + dy^2),$$

establishing, in particular, that $g_f(\delta_x, \delta_y) = 0$, so that, if $A^\xi \delta_x \in \langle \delta_x \rangle$, for either unit $\xi \in \Gamma(N_f)$, then equation (8.1) is established. It is obvious that, conversely, equation (8.1) forces, in particular,

$$A^\xi \delta_x \in \langle \delta_x \rangle, \;\; A^\xi \delta_y \in \langle \delta_y \rangle,$$

for either unit normal vector field ξ to f. Classical isothermic surfaces extend naturally to immersions $f : (M, g_f) \to \bar{M}$, into a Riemannian manifold $\bar{M}$, admitting, at every point, conformal coordinates which diagonalize the second fundamental form,[1] which we still refer to as conformal curvature line coordinates. Equation (2.6) makes clear that the isothermic condition is a conformal invariant (even though the second fundamental form is not), having in consideration that under a conformal change of metric in $\bar{M}$, the metric induced in M changes conformally. In fact, it makes clear, furthermore, that conformal curvature line coordinates are preserved under conformal changes of the metric. Hence, as very well-known:

Theorem 8.1. *Conformal curvature line coordinates are preserved by conformal diffeomorphisms.*

Theorem 8.1 establishes, in particular, the Möbius invariance of the class of isothermic surfaces. We define $\Lambda : (M, \mathcal{C}_\Lambda) \to (\mathbb{P}(\mathcal{L}), \mathcal{C}_{\mathbb{P}(\mathcal{L})})$ to be an *isothermic surface* if, fixing $h \in \mathcal{C}_{\mathbb{P}(\mathcal{L})}$ (independently of the choice of h), $\Lambda : (M, \Lambda^* h) \to (\mathbb{P}(\mathcal{L}), h)$ is an isothermic surface, with $\Lambda^* h$ denoting the metric induced in M by Λ from h. We formulate it next following a manifestly conformally invariant characterization of isothermic surfaces established by F. Burstall and U. Pinkall.

Definition 8.2. *Λ is said to be an* isothermic surface *if there exists a non-zero closed real 1-form η with values in $\Lambda \wedge \Lambda^\perp$.*

This formulation of isothermic surfaces is discussed in [**13**], [**11**], [**53**] and [**32**]. For the relationship between this formulation and the classical one, see [**13**], [**53**] and [**32**] (see, in particular, §5.3.19).[2]

[1] Or, equivalently, which diagonalize simultaneously all shape operators (the verification presented above for the particular case $\bar{M} = \mathbb{R}^3$ clearly holds for general $\bar{M}$).

[2] Without wishing to go into detail, it is worth remarking on this relationship. (For more details, see [**10**] and [**53**].) Given $f, f^c : M \to \mathbb{R}^n$ immersions of M in Euclidean n-space, f and f^c are said to be *Christoffel transforms* of each other, or *dual isothermic surfaces*, if f and f^c have parallel tangent

Constant mean curvature surfaces in 3-space are well-known examples of isothermic surfaces (see Section 8.2), as well as surfaces of revolution, cones and cylinders (see, for example, [**53**]).

Under the conditions of Definition 8.2, we may, alternatively, refer to the isothermic surface Λ as the pair (Λ, η). As Λ is not contained in any 2-sphere (cf. (1.13)), such η is unique up to non-zero constant real scale, cf. [**53**] (see, in particular, Proposition 1.25). It is very useful to know that, cf. [**53**] (see, in particular, Proposition 1.11):

Lemma 8.3. *If (Λ, η) is an isothermic surface, then $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$.*

It follows, in particular, that:

Lemma 8.4. *If (Λ, η) is an isothermic surface, then*

$$d^{\mathcal{D}}\eta = 0 = [\mathcal{N} \wedge \eta]. \tag{8.2}$$

Proof. Suppose (Λ, η) is isothermic. Then, in particular, the form η is closed, $d^{\mathcal{D}}\eta + [\mathcal{N} \wedge \eta] = 0$. On the other hand, according to Lemma 8.3, $\eta \in \Omega^1(\wedge^2 S)$. Hence, by the $\mathcal{D}$-parallelness of S and $S^\perp$, $d^{\mathcal{D}}\eta \in \Omega^2(\wedge^2 S \oplus \wedge^2 S^\perp)$; whereas, as $\mathcal{N}$ takes values in $S \wedge S^\perp$, $[\mathcal{N} \wedge \eta] \in \Omega^2(S \wedge S^\perp)$. We conclude that $d^{\mathcal{D}}\eta$ and $[\mathcal{N} \wedge \eta]$ vanish separately. □

Remark 8.5. *Following equation* (2.17)*, we have*

$$[(\Lambda \wedge \Lambda^\perp) \wedge (\Lambda \wedge \Lambda^\perp)] \subset \Lambda \wedge \Lambda = \{0\} \tag{8.3}$$

and, therefore, given $\eta \in \Omega^1(\Lambda \wedge \Lambda^\perp)$,

$$[\eta \wedge \eta] = 0. \tag{8.4}$$

Theorem 8.1 establishes, in particular:

Theorem 8.6. *Λ is isothermic if and only if, fixing $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero, so is the surface $\sigma_\infty : M \to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. Furthermore: Λ shares conformal curvature line coordinates with σ_{v_∞}, for all v_∞.*

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planes, i.e., $df(TM) = df^c(TM)$; f and f^c induce the same conformal structure on M; and f and f^c induce opposite orientations on M, i.e., $df^c \circ df^{-1} : df(TM) \to df(TM)$ has negative determinant. A result by Christoffel [**20**], for $n = 3$, and B. Palmer [**49**], for arbitrary n, characterizes isothermic surfaces immersed in $\mathbb{R}^n$ by the existence of a dual isothermic surface. Since $n \geq 3$, we can choose $v_\infty \in \mathcal{L}$ such that $\Lambda_p \neq \langle v_\infty \rangle$, for all $p \in M$, and to define then a surface $\sigma_{0,\infty}$ in Euclidean n-space by stereographic projection of pole $x_0 \in S^n$ of $\Lambda_p \in \mathbb{P}(\mathcal{L})\backslash\{\langle v_\infty \rangle\} = S^n \backslash \{x_0\}$, for each $p \in M$. According to Theorem 8.1, Λ is isothermic if and only if so is $\sigma_{0,\infty}$. Let σ_∞ be the surface defined by Λ in S_{v_∞}. One verifies that, if $\sigma_{0,\infty}$ is isothermic and $\sigma^c_{0,\infty}$ is a dual isothermic surface to $\sigma_{0,\infty}$, then

$$\eta := \sigma_\infty \wedge (d\sigma^c_{0,\infty} + (\sigma_{0,\infty}, d\sigma^c_{0,\infty})v_\infty)$$

is a form in the conditions of Definition 8.2; and, conversely, that the existence of a form η such that (Λ, η) is isothermic establishes the existence of a dual isothermic surface to $\sigma_{0,\infty}$.

8.1.2. Isothermic condition and Hopf differential. The Hopf differential is closely related to $\Pi^{(2,0)} := \Pi|_{T^{1,0}M\times T^{1,0}M}$, giving rise to yet another characterization of isothermic surfaces in space-forms, which we present in this section.

Fix a non-zero v_∞ in $\mathbb{R}^{n+1,1}$ and consider the surface $\sigma_\infty : M \to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. Recall that the pull-back bundle by σ_∞ of the tangent bundle TS_{v_∞} consists of the orthogonal complement in $\underline{\mathbb{R}}^{n+1,1}$ of the non-degenerate bundle $\langle\sigma_\infty, v_\infty\rangle$, $\sigma_\infty^* TS_{v_\infty} = \langle\sigma_\infty, v_\infty\rangle^\perp$. Let π_{N_∞} denote the orthogonal projection of $\underline{\mathbb{R}}^{n+1,1} = d\sigma_\infty(TM) \oplus N_\infty \oplus \langle v_\infty, \sigma_\infty\rangle$ onto N_∞. Fix a holomorphic chart $z = x + iy$ of $(M, \mathcal{C}_\Lambda)$. Observe that

$$(\sigma_\infty)_{zz} \in \Gamma(\sigma_\infty^* TS_{v_\infty}).$$

Indeed, differentiation of $(\sigma_\infty, v_\infty) = -1$ shows that $((\sigma_\infty)_z, v_\infty) = 0$ and, consequently, $((\sigma_\infty)_{zz}, v_\infty) = 0$; whereas differentiation of $(\sigma_\infty, (\sigma_\infty)_z) = 0$ shows that $(\sigma_\infty)_z$ is orthogonal to σ_∞. It follows that $(\sigma_\infty)_{zz} - \pi_{N_\infty}(\sigma_\infty)_{zz} \in \Gamma(d\sigma_\infty(TM)) \subset \Gamma(S)$ and, therefore, $(\sigma_\infty)_{zz} - \pi_{N_\infty}(\sigma_\infty)_{zz} - (\pi_{N_\infty}(\sigma_\infty)_{zz}, \mathcal{H}_\infty)\sigma_\infty \in \Gamma(S)$. We conclude that

$$\pi_{S^\perp}(\sigma_\infty)_{zz} = \mathcal{Q}(\pi_{N_\infty}(\sigma_\infty)_{zz}), \tag{8.5}$$

for the isomorphism $\mathcal{Q} : N_\infty \to S^\perp$ defined in Section 2.2. Now write $\sigma^z = \lambda\sigma_\infty$ with $\lambda \in \Gamma(\underline{\mathbb{R}})$ never-zero. Then $\sigma^z_{zz} = \lambda(\sigma_\infty)_{zz} + 2\lambda_z(\sigma_\infty)_z + \lambda_{zz}\sigma_\infty$ and, therefore,

$$k^z = \lambda\, \pi_{S^\perp}(\sigma_\infty)_{zz}. \tag{8.6}$$

It follows that, under the isomorphism $\mathcal{Q}$, the Hopf differential k^z is a real scale of

$$\pi_{N_\infty}(\sigma_\infty)_{zz} = \Pi_\infty(\delta_z, \delta_z) = \frac{1}{4}\left(\Pi_\infty(\delta_x, \delta_x) - 2i\Pi_\infty(\delta_x, \delta_y) - \Pi_\infty(\delta_y, \delta_y)\right). \tag{8.7}$$

We are led to the following characterization of isothermic surfaces in terms of the Hopf differential, presented in [**14**]:

Lemma 8.7. *The surface Λ is isothermic if and only if around each point there exists a holomorphic chart of $(M, \mathcal{C}_\Lambda)$ with respect to which the Hopf differential of Λ is a real section of $S^\perp$. Furthermore: the conformal coordinates x, y are curvature line coordinates to Λ if and only if k^z is real.*

Proof. The conformal coordinates x, y are curvature line coordinates of σ_∞ if and only if $\Pi_\infty(\delta_x, \delta_y) = 0$, or, equivalently, $\pi_{N_\infty}(\sigma_\infty)_{zz}$ is real. Since $\mathcal{Q}$ is an isomorphism of real bundles, the reality of $\pi_{N_\infty}(\sigma_\infty)_{zz}$ is equivalent to that of k^z. $\square$

8.1.3. Transformations of isothermic surfaces. Isothermic surfaces in $\mathbb{R}^3$ were studied intensively at the turn of the 20th century and a rich transformation theory of these surfaces was developed in the works of Darboux [**22**], Calapso [**16**], [**17**] and Bianchi [**2**], [**3**]. The loop group formalism provides a context in which the results of Bianchi, Calapso and Darboux can be generalized. Following the work of F. Burstall,

D. Calderbank and U. Pinkall [**11**], [**13**], we characterize isothermic surfaces by the flatness of a certain $\mathbb{R}$-family of metric connections on $\underline{\mathbb{R}}^{n+1,1}$ and define, in terms of this family of connections, both the isothermic spectral deformation, discovered in the classical setting by Calapso and, independently, by Bianchi; and the isothermic Darboux transformation.

Let η be a non-zero real 1-form with values in $\Lambda\wedge\Lambda^{\perp}$. For each $t\in\mathbb{R}$, set

$$d^t_\eta := d + t\eta,$$

defining a connection of $\underline{\mathbb{C}}^{n+2}$. The reality of η establishes that of d^t_η, whereas its skew-symmetry establishes d^t_η as a metric connection. As established in [**11**] and [**13**]:

Theorem 8.8. *(Λ,η) is isothermic if and only if d^t_η is a flat connection, for each $t\in\mathbb{R}$.*

The proof is immediate, but worth presenting:

Proof. The curvature tensor R^t of d^t_η is given by $R^t = R^d + t d\eta + \frac{t^2}{2}[\eta\wedge\eta] = 0$. Equation (8.4) makes then clear that η is closed if and only if $R^t = 0$, for all t. $\square$

One shall be aware of the ambiguity that the notation d^t_η carries, for $t=\pm 1$, with respect to the constrained Willmore spectral deformation of parameter t, corresponding to the multiplier η, in the case (Λ,η) is an isothermic surface admitting η as a multiplier.

The isothermic spectral deformation. As we verify next, if (Λ,η) is isothermic, then so is the transformation of Λ defined by the flat metric connection d^t_η, for each $t\in\mathbb{R}$. Associated to an isothermic surface, we have a one-parameter family of isothermic surfaces, discovered in the classical setting by Calapso [**16**], [**17**] and, independently, by Bianchi [**2**], [**3**].

Suppose (Λ,η) is isothermic, so that, in particular, d^t_η is a flat metric connection on $\underline{\mathbb{R}}^{n+1,1}$, for each $t\in\mathbb{R}$. Given $t\in\mathbb{R}$ and $\sigma\in\Gamma(\Lambda)$,

$$d^t_\eta\sigma = d\sigma, \tag{8.8}$$

showing that Λ is still a d^t_η-surface, or, equivalently, the transformation Λ^t_η of Λ defined by the connection d^t_η is still a surface. Furthermore:

Theorem 8.9. *Let (Λ,η) be an isothermic surface. Then, for each $t\in\mathbb{R}$, the transformation Λ^t_η of Λ defined by the flat metric connection d^t_η is still isothermic.*

Fix $t\in\mathbb{R}$ and $\phi^t_\eta : (\underline{\mathbb{R}}^{n+1,1}, d^t_\eta)\to(\underline{\mathbb{R}}^{n+1,1}, d)$ an isomorphism. The proof of the theorem will consist of showing that $(\Lambda^t_\eta, \mathrm{Ad}_{\phi^t_\eta}\eta)$ is isothermic.

PROOF. If $\eta = \sigma \wedge \mu$, with $\sigma \in \Gamma(\Lambda)$ and $\mu \in \Omega^1(\Lambda^\perp)$, then, recalling (2.18), $\mathrm{Ad}_{\phi^t_\eta}\eta = \phi^t_\eta\sigma \wedge \phi^t_\eta\mu$ is a non-zero real 1-form with values in $\Omega^1((\phi^t_\eta\Lambda) \wedge (\phi^t_\eta\Lambda)^\perp)$ and

$$d(\mathrm{Ad}_{\phi^t_\eta}\eta) = \phi^t_\eta \circ d^{d^t_\eta}\eta \circ (\phi^t_\eta)^{-1} = \phi^t_\eta \circ (d\eta + t[\eta \wedge \eta]) \circ (\phi^t_\eta)^{-1} = 0.$$

□

We may refer to Λ^t_η as the *isothermic* (t, η)*-transformation* of Λ. Note that, if (Λ, η') is isothermic, for some non-zero real 1-form η', with values in $\Lambda \wedge \Lambda^\perp$, then $\eta' = t_\eta \eta$, for some $t_\eta \in \mathbb{R}$ and the isothermic (t, η')-transformation of Λ coincides with the $(t\, t_\eta, \eta)$-transformation.

Observe that, given $t' \in \mathbb{R}$ and $\phi^{t'}_{\mathrm{Ad}_{\phi^t_\eta}\eta} : (\underline{\mathbb{R}}^{n+1,1}, d^{t'}_{\mathrm{Ad}_{\phi^t_\eta}\eta}) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ an isomorphism, $\phi^{t'}_{\mathrm{Ad}_{\phi^t_\eta}\eta}\phi^t_\eta : (\underline{\mathbb{R}}^{n+1,1}, d^{t+t'}_\eta) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ is an isomorphism, to conclude that

$$\Lambda^{t+t'}_\eta = (\Lambda^t_\eta)^{t'}_{\mathrm{Ad}_{\phi^t_\eta}\eta},$$

we have a one-parameter family of isothermic surfaces. In the particular case of Euclidean 3-space, this is the T-transform, found by Calapso [**16**], [**17**] and, independently, by Bianchi [**2**], [**3**].

We complete this section by verifying that, up to reparametrization, this isothermic spectral deformation coincides with the one presented in [**14**].[3] First of all, note that, in view of (8.8), given $\sigma \in \Gamma(\Lambda)$ never-zero,

$$g_{\phi^t_\eta\sigma} = g^{d^t_\eta}_\sigma = g_\sigma, \tag{8.9}$$

showing that the deformation defined by d^t_η preserves the conformal structure,

$$\mathcal{C}_{\Lambda^t_\eta} = \mathcal{C}_\Lambda.$$

It preserves the central sphere congruence, as well. Indeed, fixing a holomorphic chart z of $(M, \mathcal{C}_{\Lambda^t_\eta}) = (M, \mathcal{C}_\Lambda)$, we have $(d^t_\eta)_{\delta_{\bar z}}(d^t_\eta)_{\delta_z}\sigma = (\sigma)_{z\bar z} + t\eta_{\delta_{\bar z}}\sigma_z = \sigma_{z\bar z} \,\mathrm{mod}\Lambda$, in view of equation (8.8), showing that $S^{d^t_\eta} = S$ and, ultimately, according to (3.3), that

$$S_{\phi^t_\eta\Lambda} = \phi^t_\eta S. \tag{8.10}$$

According to (8.9), on the other hand, $g_{\phi^t_\eta\sigma^z} = g_{\sigma^z} = g_z$, showing that $\phi^t_\eta\sigma^z$ is the normalized section of $\phi^t_\eta\Lambda$ with respect to z. Note that, as η takes values in $\Lambda \wedge \Lambda^\perp$, $\eta\Lambda^\perp$ takes values in Λ, and define $\eta^z \in \mathbb{C}^\infty(M, \mathbb{R})$ by $\eta_{\delta_z}\sigma^z_z = \eta^z\sigma^z$. Then

$$(\phi^t_\eta\sigma^z)_{zz} = (\phi^t_\eta\sigma^z_z)_z = \phi^t_\eta(\sigma^z_{zz} + t\eta_{\delta_z}\sigma^z_z) = -\frac{1}{2}\,(c^z - 2t\eta^z)\,\phi^t_\eta\sigma^z + \phi^t_\eta k^z.$$

[3]The omission, in [**14**], of reference to the transformation rules of the Hopf differential and of the normal connection shall be understood as preservation.

We conclude that k_t^z and c_t^z, the Hopf differential and the Schwarzian derivative, respectively, of $\phi_\eta^t\Lambda$ with respect to z, relate to those of Λ by

$$k_t^z = \phi_\eta^t k^z, \quad c_t^z = c^z - 2t\eta^z.$$

By Lemma 5.5, having in consideration (8.10), the conclusion follows.

Isothermic Darboux transformation. Darboux [**22**] discovered a transformation of isothermic surfaces in $\mathbb{R}^3$: the surface and its Darboux transform are characterized by being conformal and curvature line preserving and enveloping some 2-sphere congruence. In this section, we present a manifestly conformally invariant formulation of Darboux transforms of isothermic surfaces, due to F. Burstall and U. Pinkall, in terms of the family of flat metric connections, presented above, characterizing the isothermic condition. For further reference, we make a very brief description of the Darboux transformation of isothermic surfaces in Euclidean n-space via solutions of a Ricatti equation, presented in [**10**] as a direct extension of the case $n = 3, 4$, discovered by Hertrich-Jeromin−Pedit [**33**].

In [**33**], U. Hertrich-Jeromin and F. Pedit develop isothermic surface theory in Euclidean n-space, for $n = 3, 4$. In particular, they define Darboux transformation, based on the solution of a Ricatti equation, that, when restricted to codimension 1, becomes classical. In [**10**], F. Burstall presents a direct extension to general n of this approach to isothermic surfaces and Darboux transformation. For further reference, we describe it here very briefly (for more details, see [**10**]). The starting point is the fact that an immersion $f : M \to \mathbb{R}^n$ is isothermic if and only if there exists another immersion $f^c : M \to \mathbb{R}^n$ such that[4]

$$df \wedge df^c = 0, \tag{8.11}$$

where we multiply the coefficients of these $\mathbb{R}^n$-valued 1-forms using the product of the Clifford algebra $\mathcal{C}l_n$ of $\mathbb{R}^n$. Equation (8.11) is the integrability condition for a Ricatti equation involving an auxiliary parameter $r \in \mathbb{R}\backslash\{0\}$:

$$dg = rgdf^cg - df,$$

where again all multiplications take place in $\mathcal{C}l_n$. We construct a new isothermic surface $\hat{f}$ by setting

$$\hat{f} = f + g$$

and verify that, just as in the classical case, f and $\hat{f}$ are characterized by the conditions that they have the same conformal structure and curvature lines and are enveloping

[4]Equation (8.11) characterizes f^c as a Christoffel transform of f.

surfaces of a 2-sphere congruence. We refer to $\hat{f}$ as the *Darboux transform of* f *of parameters* r, g.

F. Burstall and U. Pinkall generalized the Darboux transformation to isothermic surfaces in general space-form as follows, in a manifestly conformally invariant formulation:

Definition 8.10. *Let* (Λ, η) *be an isothermic surface. A surface* $\hat{\Lambda}$ *is an* isothermic Darboux transform *of* Λ *if* $\hat{\Lambda} \cap \Lambda = \{0\}$ *and there is a non-zero real constant* m *for which* $\hat{\Lambda}$ *is* $(d + m\eta)$*-parallel.*

This formulation is discussed in [**13**], [**32**] (see, in particular, §5.4.8), [**53**] and [**11**]. For the relationship between this approach to isothermic Darboux transformation and the classical one, see [**53**].

8.1.4. Isothermic condition and uniqueness of multiplier. In this section, we characterize isothermic constrained Willmore surfaces in space-forms by the non-uniqueness of multiplier and establish the set of multipliers to an isothermic q-constrained Willmore surface (Λ, η) as the 1-dimensional affine space $q + \langle *\eta \rangle_{\mathbb{R}}$.

Theorem 8.11. *Suppose* Λ *is a constrained Willmore surface. The uniqueness of multiplier to* Λ *is equivalent to* Λ *being not isothermic.*

Proof. Suppose Λ is a constrained Willmore surface and $q_1 \neq q_2$ are multipliers to Λ. Set $\eta := *(q_1 - q_2)$, defining, in this way, a non-zero real 1-form with values in $\Lambda \wedge \Lambda^{(1)} \subset \Lambda \wedge \Lambda^{\perp}$. The fact that $d^{\mathcal{D}} q_1 = 0 = d^{\mathcal{D}} q_2$ establishes $d^{\mathcal{D}} * \eta = 0$, or, equivalently, cf. Lemma 5.10, $d^{\mathcal{D}} \eta = 0$; whereas $[q_1 \wedge *\mathcal{N}] = \frac{1}{2} d^{\mathcal{D}} * \mathcal{N} = [q_2 \wedge *\mathcal{N}]$ gives $[\mathcal{N} \wedge \eta] = [*\eta \wedge *\mathcal{N}] = [(q_2 - q_1) \wedge *\mathcal{N}] = 0$, recalling equation (2.21). We conclude that η is closed and, therefore, that (Λ, η) is isothermic.

Conversely, suppose (Λ, η) is an isothermic q-constrained Willmore surface, for some $\eta \in \Omega^1(\Lambda \wedge \Lambda^{\perp})$ and $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. Recalling Lemma 8.3, set $q' := q + *\eta$, defining a real 1-form $q' \neq q$ with values in $\Lambda \wedge \Lambda^{(1)}$. Cf. Lemma 8.4, $d^{\mathcal{D}} \eta = 0 = [\mathcal{N} \wedge \eta]$. Equivalently (recall Lemma 5.10), $d^{\mathcal{D}} * \eta = 0 = [*\eta \wedge *\mathcal{N}]$. Hence $d^{\mathcal{D}} q' = d^{\mathcal{D}} q = 0$ and $2[q' \wedge *\mathcal{N}] = 2[q \wedge *\mathcal{N}] = d^{\mathcal{D}} * \mathcal{N}$, showing that q' is a multiplier to Λ, as well as q, and completing the proof. □

Constant mean curvature surfaces in 3-space are examples of isothermic constrained Willmore surfaces, as proven by J. Richter [**51**]. A classical result by Thomsen [**55**] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form, showing that the zero multiplier is not necessarily the only multiplier to a constrained Willmore surface with no constraint on the conformal structure.

Suppose (Λ, η) is isothermic. Analogously to what was observed in the proof of Theorem 8.11 for the particular case $t = 1$, we verify, that, if q is a multiplier to Λ, then so is

$$q^t := q + t * \eta,$$

for each $t \in \mathbb{R}$. In the proof of Theorem 8.11, we have, on the other hand, verified that, if q_1 and q_2 are distinct multipliers to Λ, then $(\Lambda, *(q_1 - q_2))$ is isothermic and, therefore, $q_2 = q_1 + t * \eta$, for some (non-zero) constant $t \in \mathbb{R}$. We conclude that, if Λ is constrained Willmore and q is a multiplier to Λ, then the set of multipliers to Λ is the affine space $q + \langle *\eta \rangle_{\mathbb{R}}$. In particular, the set of multipliers to an isothermic Willmore surface (Λ, η) consists of the 1-dimensional vector space $\langle *\eta \rangle_{\mathbb{R}}$.

8.1.5. Isothermic condition under constrained Willmore transformation. The constrained Willmore spectral deformation is known to preserve the isothermic condition, cf. [**14**]. Next we derive it in our setting.

Theorem 8.12. *The constrained Willmore spectral deformation preserves the isothermic condition.*

The proof of the theorem will consist of showing that, if (Λ, η) is an isothermic q-constrained Willmore surface, for some $\eta \in \Omega^1(\Lambda \wedge \Lambda^\perp)$ and $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$, then, fixing $\lambda \in S^1$ and an isomorphism $\phi_q^\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, d)$, $(\phi_q^\lambda \Lambda, \mathrm{Ad}_{\phi_q^\lambda} \eta_\lambda)$ is isothermic, for

$$\eta_\lambda := \lambda^{-1} \eta^{1,0} + \lambda\, \eta^{0,1}.$$

Proof. Suppose (Λ, η) is an isothermic q-constrained Willmore surface, for some $\eta \in \Omega^1(\Lambda \wedge \Lambda^\perp)$ and $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. Fix $\lambda \in S^1$ and $\phi_q^\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ an isomorphism. Write $\eta = \sigma \wedge \mu$, with $\sigma \in \Gamma(\Lambda)$ and $\mu \in \Omega^1(\Lambda^\perp)$. Then $\mathrm{Ad}_{\phi_q^\lambda} \eta_\lambda = \phi_q^\lambda \sigma \wedge \phi_q^\lambda (\lambda^{-1} \mu^{1,0} + \lambda \mu^{0,1})$ is a non-zero real 1-form with values in $(\phi_q^\lambda \Lambda) \wedge (\phi_q^\lambda \Lambda)^\perp$ and $d(\mathrm{Ad}_{\phi_q^\lambda} \eta_\lambda) = \phi_q^\lambda \circ d^{d_q^\lambda} \eta_\lambda \circ (\phi_q^\lambda)^{-1}$ vanishes if and only if

$$d^{d_q^\lambda} \eta_\lambda = d^{\mathcal{D}} \eta_\lambda + [(\lambda^{-1} \mathcal{N}^{1,0} + \lambda \mathcal{N}^{0,1} + (\lambda^{-2} - 1) q^{1,0} + (\lambda^2 - 1) q^{0,1}) \wedge \eta_\lambda]$$

does. Note that $q \in \Omega^1(\Lambda \wedge \Lambda^\perp)$. Thus, by (8.3), $[q^{1,0} \wedge \eta^{0,1}] = 0 = [q^{0,1} \wedge \eta^{1,0}]$ and, therefore, $d^{d_q^\lambda} \eta_\lambda = d^{\mathcal{D}} \eta_\lambda + [\mathcal{N} \wedge \eta]$. Lemma 8.4, together with Lemma 5.10 (having in consideration Lemma 8.3), establishes $d^{\mathcal{D}} \eta^{1,0} = 0 = d^{\mathcal{D}} \eta^{0,1}$, $[\mathcal{N} \wedge \eta] = 0$, completing the proof. $\square$

As for Bäcklund transformation of constrained Willmore surfaces, we believe it does not necessarily preserve the isothermic condition. This shall be the subject of further work.

8.2. Constant mean curvature surfaces in 3-space

Minimal surfaces arose originally as surfaces that minimized the surface area, subject to some constraint, such as total volume enclosed. Physical processes which can be modeled by minimal surfaces include the formation of soap bubbles. A soap bubble can can be thought of as an excellent approximation of some ideal elastic matter, which encloses a volume and exists in an equilibrium where slightly greater pressure inside the bubble is balanced by the area-minimizing forces of the bubble itself. Minimal surfaces are defined as surfaces with zero mean curvature and can be extended to surfaces with constant, not necessarily zero, mean curvature. Constant mean curvature surfaces in 3-dimensional space-forms form a very important class of isothermic constrained Willmore surfaces, as proven by J. Richter [**51**], with constrained Willmore Bäcklund transformations; both constrained Willmore and isothermic spectral deformations; as well as a spectral deformation of their own and, in the Euclidean case, isothermic Darboux transformations and Bianchi-Bäcklund transformations. The isothermic spectral deformation is known to preserve the constancy of the mean curvature of a surface in some space-form, cf. [**14**]. Characterized as the class of constrained Willmore surfaces in 3-dimensional space-forms admitting a conserved quantity, the class of CMC surfaces in 3-space is known to be preserved by both constrained Willmore spectral deformation and Bäcklund transformation, for special choices of parameters. We verify that both the space-form and the mean curvature are preserved by constrained Willmore Bäcklund transformation and investigate how these change under constrained Willmore and isothermic spectral deformation. We present the classical CMC spectral deformation by means of the action of a loop of flat metric connections on the class of CMC surfaces in 3-space (preserving the space-form and the mean curvature) and observe that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation. These spectral deformations of CMC surfaces in 3-space are, in this way, all closely related and, therefore, closely related to constrained Willmore Bäcklund transformation. S. Kobayashi and J.-I. Inoguchi [**35**] proved that isothermic Darboux transformation of CMC surfaces in Euclidean 3-space is equivalent to Bianchi-Bäcklund transformation. We believe isothermic Darboux transformation of a CMC surface in Euclidean 3-space can be obtained as a particular case of constrained Willmore Bäcklund transformation. This shall be the subject of further work. In contrast to isothermic or constrained Willmore surfaces in space-forms, surfaces of constant mean curvature are not conformally invariant objects.

Throughout this section, consider $n = 3$. For simplicity, we use T and $\perp$ to indicate the orthogonal projections of $\underline{\mathbb{R}}^{4,1}$ onto S and $S^{\perp}$, respectively.

Remark 8.13. *In contrast to isothermic or constrained Willmore surfaces in space-forms, surfaces of constant mean curvature are not conformally invariant objects (recall equation* (2.7)*).*

Fix $v_\infty \in \mathbb{R}^{4,1}$ non-zero. Consider the surface $\sigma_\infty : M \to S_{v_\infty}$, in the space-form S_{v_∞}, defined by Λ. Given $\xi \in \Gamma(N_\infty)$ unit, the mean curvature H^ξ_∞ of σ_∞ with respect to ξ is given by $H^\xi_\infty = (\xi, \mathcal{H}_\infty) = -(\xi + (\xi, \mathcal{H}_\infty)\sigma_\infty, v_\infty) = -(\mathcal{Q}\xi, v^\perp_\infty)$, for $\mathcal{Q} : N_\infty \to S^\perp$ the isometry defined in Section 2.2. Because we are in codimension 1, $S^\perp = \langle \mathcal{Q}\xi \rangle$ and we conclude that $H^\xi_\infty = \pm(v^\perp_\infty, v^\perp_\infty)^{\frac{1}{2}}$, depending on the sign of ξ. We define the *mean curvature of* Λ *in the space-form* S_{v_∞} to be

$$H_\infty := (v^\perp_\infty, v^\perp_\infty)^{\frac{1}{2}}$$

and define Λ to be a *constant mean curvature surface* (respectively, a *minimal surface*) *in the space-form* S_{v_∞} if σ_∞ is so:

Definition 8.14. Λ *is said to be a* constant mean curvature *(CMC)* surface *in the space-form* S_{v_∞} *if* $(v^\perp_\infty, v^\perp_\infty)$ *is constant. In the case* $v_\infty \in \Gamma(S)$, Λ *is said to be, specifically, a* minimal surface *in* S_{v_∞}.

Let N be the real unit section of $S^\perp$ for which

$$v^\perp_\infty = H_\infty N, \tag{8.12}$$

which, in the particular case Λ is minimal in S_{v_∞}, is defined only up to sign.[5]

Remark 8.15. *If* $v^T_\infty = 0$*, then* $v^\perp_\infty$ *is a constant section of* $S^\perp$*, and so is then* $H^{-1}_\infty v^\perp_\infty = N$*. The constancy of the normal to* S*, in its turn, establishes the constancy of* S*, establishing, in particular, that* Λ *lies in a* 2*-sphere:* $\mathbb{P}(\mathcal{L} \cap S)$*; which contradicts* (1.13)*. Thus*

$$v^T_\infty \neq 0. \tag{8.13}$$

It is useful to note that, in view of the constancy of (N, N), and because we are in codimension 1 (and, therefore, $S^\perp = \langle N \rangle$), we have

$$dN \in \Omega^1(S). \tag{8.14}$$

8.2.1. CMC surfaces in 3-space as isothermic constrained Willmore surfaces with a conserved quantity. Constant mean curvature surfaces in 3-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven

[5]There are exactly two possible choices of real unit sections of $S^\perp$, symmetrical of each other - they are $\mathcal{Q}\xi$ with ξ a unit normal vector field to σ_∞. Unless $H_\infty = 0$, condition (8.12) determines N as $\mathcal{Q}\xi$ for ξ the unit normal vector field to σ_∞ with $H^\xi_\infty = -H_\infty$ (or, equivalently, $H^\xi_\infty < 0$).

by J. Richter [**51**]. In this section, we establish it in our setting. We present a 1-form, derived[6] by F. Burstall and D. Calderbank from a surface with constant mean curvature in 3-space, which establishes the surface as an isothermic surface and for which scaling by the mean curvature provides a multiplier to the surface. We prove also that constant mean curvature surfaces in 3-dimensional space-forms are the constrained Willmore surfaces in 3-space admitting a conserved quantity.

Suppose Λ has constant mean curvature H_∞ in S_{v_∞}. Set

$$\eta_\infty := \frac{1}{2}\,\sigma_\infty \wedge dN.$$

We may, alternatively, use η_∞^N to denote η_∞, in order to avoid the ambiguity with respect to the sign of N in the particular case $H_\infty = 0$.

Theorem 8.16. (Λ, η_∞) *is isothermic.*

Proof. The reality of η_∞ is immediate, in view of the reality of both σ_∞ and N. Fix a holomorphic chart z of M. The fact that N is orthogonal to S shows, in particular, that $(N_z, \sigma_\infty) = 0 = (N_z, (\sigma_\infty)_{\bar z})$ and, therefore, in view of the maximal isotropy of $\Lambda^{0,1}$ in S, that $N_z \in \Gamma(\Lambda^{0,1})$. On the other hand, differentiation of $(\sigma_\infty, v_\infty) = -1$ shows that $((\sigma_\infty)_{\bar z}, v_\infty) = 0$ and, consequently, that the component of N_z with respect to σ_∞ in the frame $(\sigma_\infty, (\sigma_\infty)_{\bar z})$ of $\Lambda^{0,1}$ is $-(N_z, v_\infty) = -(N, v_\infty^\perp)_z = (H_\infty)_z = 0$. Thus, having in consideration the reality of N,

$$N_z \in \Gamma\langle(\sigma_\infty)_{\bar z}\rangle, \quad N_{\bar z} \in \Gamma\langle(\sigma_\infty)_z\rangle. \tag{8.15}$$

In particular, $dN \in \Omega^1(\Lambda^\perp)$ and, therefore, $\eta_\infty \in \Omega^1(\Lambda \wedge \Lambda^\perp)$.[7] It follows from (8.15), on the other hand, having in consideration (1.9), that

$$d(\sigma_\infty \wedge dN)(\delta_z, \delta_{\bar z}) = (\sigma_\infty)_z \wedge N_{\bar z} - (\sigma_\infty)_{\bar z} \wedge N_z + \sigma_\infty \wedge (N_{z\bar z} - N_{\bar z z}) = 0,$$

or, equivalently, $d\eta_\infty = 0$. Lastly, note that, if $\sigma_\infty \wedge dN = 0$, or, equivalently, $dN \in \Omega^1(\langle\sigma_\infty\rangle)$, then, in particular, $(N_{\bar z}, (\sigma_\infty)_{\bar z}) = 0$. Together with (8.15), and having in consideration that $((\sigma_\infty)_z, (\sigma_\infty)_{\bar z})$ is never-zero, this forces $N_{\bar z}$ to vanish and, therefore, in view of the reality of N, N to be constant. But, as observed in Remark 8.15, the constancy of N forces Λ to lie in a 2-sphere, which is not the case. Hence $\sigma_\infty \wedge dN$ is non-zero, completing the proof. $\square$

Remark 8.17. *In the proof of Theorem 8.16, we have observed, in particular, that* $N_z \in \Gamma(\Lambda^{0,1})$ *and, therefore,*

$$\eta_\infty^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1}), \quad \eta_\infty^{0,1} \in \Omega^{0,1}(\Lambda \wedge \Lambda^{1,0}), \tag{8.16}$$

[6] From the notion of *conserved quantity of an isothermic surface*, presented in [**53**], similarly to how a q-conserved quantity of a constrained Willmore surface determines q (cf. Remark 7.4).

[7] Furthermore, $\eta_\infty \in \Omega^1(\Lambda \wedge (\Lambda^\perp \cap S)) \subset \Omega^1(\Lambda \wedge \Lambda^{(1)})$.

in view of the reality of η_∞.

Set

$$q_\infty = H_\infty \eta_\infty.$$

Theorem 8.18. Λ *is a* q_∞*-constrained Willmore surface.*

The proof of the theorem will follow a few considerations. First note that the constancy of $(v_\infty^\perp, v_\infty^\perp)$ ensures, in particular, that $v_\infty^\perp$ is either zero or never-zero, and, on the other hand, that

$$((dv_\infty^\perp)^\perp, v_\infty^\perp) = 0. \tag{8.17}$$

In the case $v_\infty^\perp$ is never-zero, and, therefore, $S^\perp = \langle v_\infty^\perp \rangle$, equation (8.17) establishes

$$(dv_\infty^\perp)^\perp = 0, \tag{8.18}$$

an equality that, obviously, still holds in the case $v_\infty \in \Gamma(S)$. Equivalently,

$$\mathcal{N} v_\infty^\perp = dv_\infty^\perp. \tag{8.19}$$

Since $dv_\infty^\perp = H_\infty dN$, we conclude that

$$q_\infty = \frac{1}{2}\, \sigma_\infty \wedge \mathcal{N} v_\infty^\perp. \tag{8.20}$$

Now we proceed to the proof of Theorem 8.18.

Proof. According to Theorem 8.16, (Λ, η_∞) is isothermic, which, according to Lemma 8.3, establishes that the 1-form q_∞ takes values in $\Lambda \wedge \Lambda^{(1)}$. The reality of q_∞ is obviously equivalent to that of η_∞. On the other hand, by (8.2),

$$d^{\mathcal{D}} q_\infty = H_\infty\, d^{\mathcal{D}} \eta_\infty = 0.$$

To complete the proof, we are left to verify that $d^{\mathcal{D}} * \mathcal{N} = 2[q_\infty \wedge *\mathcal{N}]$.[8]

Notation: given $\Psi \in \Omega^1(\mathrm{End}(\underline{\mathbb{R}}^{4,1}))$ and $\psi \in \Omega^1(\underline{\mathbb{R}}^{4,1})$, $[\Psi, \psi]$ denotes the 2-form with values in $\underline{\mathbb{R}}^{4,1}$ defined by $[\Psi, \psi](X, Y) := \Psi_X \psi_Y - \Psi_Y \psi_X$, for $X, Y \in \Gamma(TM)$.

In view of (4.19),

$$(d^{\mathcal{D}} * \mathcal{N}) v_\infty^T = -2i\, (d^{\mathcal{D}} \mathcal{N}^{1,0}) v_\infty^T = -2i\, d^{\mathcal{D}} (\mathcal{N}^{1,0} v_\infty^T) - 2i[\mathcal{N}^{1,0}, \mathcal{D}^{0,1} v_\infty^T].$$

On the other hand, by the constancy of the section v_∞ of $\underline{\mathbb{R}}^{4,1}$,

$$\mathcal{D} v_\infty^T + \mathcal{D} v_\infty^\perp + \mathcal{N} v_\infty^T + \mathcal{N} v_\infty^\perp = 0, \tag{8.21}$$

and, in particular, considering the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto $S^\perp$,

$$\mathcal{N} v_\infty^T = -\mathcal{D} v_\infty^\perp. \tag{8.22}$$

[8]The scaling of η_∞ by H_∞ in order to obtain a multiplier is determined by this equation.

But, according to (8.18),

$$\mathcal{D}v_\infty^\perp = 0. \tag{8.23}$$

Hence

$$(d^{\mathcal{D}} * \mathcal{N})\, v_\infty^T = -2i\, [\mathcal{N}^{1,0}, \mathcal{D}^{0,1} v_\infty^T].$$

On the other hand,

$$2[q_\infty \wedge *\mathcal{N}]\, v_\infty^T = -2i[q_\infty \wedge \mathcal{N}^{1,0}]\, v_\infty^T + 2i[q_\infty \wedge \mathcal{N}^{0,1}]\, v_\infty^T.$$

The fact that q_∞ takes values in $\Lambda \wedge \Lambda^{(1)}$ establishes, in particular, $q_\infty\, S^\perp = 0$ and, therefore,

$$[q_\infty \wedge \mathcal{N}^{1,0}]\, v_\infty^T = [\mathcal{N}^{1,0}, q_\infty^{0,1} v_\infty^T].$$

In view of (8.20),

$$2\, q_\infty^{0,1} v_\infty^T = (\sigma_\infty \wedge \mathcal{N}^{0,1} v_\infty^\perp) v_\infty^T = -\mathcal{N}^{0,1} v_\infty^\perp - (\mathcal{N}^{0,1} v_\infty^\perp, v_\infty^T)\sigma_\infty,$$

and, therefore,

$$2\, [q_\infty \wedge \mathcal{N}^{1,0}]\, v_\infty^T = -[\mathcal{N}^{1,0}, \mathcal{N}^{0,1} v_\infty^\perp].$$

Similarly,

$$2\, [q_\infty \wedge \mathcal{N}^{0,1}]\, v_\infty^T = -[\mathcal{N}^{0,1}, \mathcal{N}^{1,0} v_\infty^\perp].$$

Now observe, in view of equation (8.18), that, given $X, Y \in \Gamma(TM)$,

$$\mathcal{N}_X \mathcal{N}_Y v_\infty^\perp - \mathcal{N}_Y \mathcal{N}_X v_\infty^\perp = \pi_{S^\perp}(d_X d_Y v_\infty^\perp - d_Y d_X v_\infty^\perp) = \pi_{S^\perp}(d_{[X,Y]} v_\infty^\perp) = 0.$$

In particular,

$$[\mathcal{N}^{1,0}, \mathcal{N}^{0,1} v_\infty^\perp] = -[\mathcal{N}^{0,1}, \mathcal{N}^{1,0} v_\infty^\perp].$$

Hence

$$2[q_\infty \wedge *\mathcal{N}]\, v_\infty^T = 2i\, [\mathcal{N}^{1,0}, \mathcal{N}^{0,1} v_\infty^\perp].$$

Going back to equation (8.21), and considering, this time, the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto S, establishes

$$\mathcal{N} v_\infty^\perp = -\mathcal{D} v_\infty^T, \tag{8.24}$$

and, ultimately,

$$(d^{\mathcal{D}} * \mathcal{N})\, v_\infty^T = 2\, [q_\infty \wedge *\mathcal{N}]\, v_\infty^T,$$

which, in view of the fact that $(\sigma_\infty, v_\infty^T) = (\sigma_\infty, v_\infty)(= -1)$ is never-zero, completes the proof, cf. Remark 5.7. $\square$

Note that, if Λ is minimal in S_{v_∞} (i.e., $H_\infty = 0$), then $q_\infty = 0$, which, according to Theorem 8.18, establishes Λ as a Willmore surface.

Corollary 8.19. *A CMC surface in 3-space is, in particular, a constrained Willmore surface. A minimal surface in 3-space is, in particular, a Willmore surface.*

Minimal surfaces in 3-space are, in particular, isothermic Willmore surfaces. Furthermore, a classical result by Thomsen [**55**] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. Hence a CMC surface in a 3-dimensional space-form is a Willmore surface if and only if it is minimal.

Next we establish a conserved quantity of a CMC surface in 3-space (see also Proposition 8.23).

Proposition 8.20. Λ *admits*

$$p_\infty(\lambda) := \lambda^{-1}\frac{1}{2}\,v_\infty^\perp + v_\infty^T + \lambda\frac{1}{2}\,v_\infty^\perp$$

as a q_∞*-conserved quantity.*

Proof. According to equations (8.22) and (8.23), we have $\mathcal{N}v_\infty^T = 0$, so that $(\mathcal{N}v_\infty^\perp, v_\infty^T) = -(v_\infty^\perp, \mathcal{N}v_\infty^T) = 0$ and, consequently, by (8.20),

$$q_\infty^{1,0}v_\infty^T = \frac{1}{2}\,(-\mathcal{N}^{1,0}v_\infty^\perp - (\mathcal{N}^{1,0}v_\infty^\perp, v_\infty^T)\sigma_\infty) = -\frac{1}{2}\,\mathcal{N}^{1,0}v_\infty^\perp. \tag{8.25}$$

Equation (8.23) completes the proof, according to Theorem 7.2. $\square$

Proposition 8.21. *A constrained Willmore surface in* 3*-space admitting a conserved quantity* $p(\lambda)$ *is, in particular, a CMC surface in the space-form* $S_{p(1)}$*.*

Proof. Let $\hat\Lambda \subset \underline{\mathbb{R}}^{4,1}$ be a constrained Willmore surface in the projectivized light-cone. Let $\perp$ indicate, temporarily, the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto the normal bundle to the central sphere congruence of $\hat\Lambda$. The existence of a conserved quantity $p(\lambda)$ of $\hat\Lambda$ establishes, in particular, cf. Theorem 7.2, the constancy of $\hat v_\infty := p(1)$. Furthermore, by (8.19), $d(\hat v_\infty^\perp, \hat v_\infty^\perp) = 2(d\hat v_\infty^\perp, \hat v_\infty^\perp) = (\mathcal{N}_{\hat\Lambda}\hat v_\infty^\perp, \hat v_\infty^\perp) = 0$, establishing $\hat\Lambda$ as a CMC surface in the space-form $S_{\hat v_\infty}$. $\square$

Theorem 8.18 combines with Propositions 8.20 and 8.21 to establish, in particular, the following:

Theorem 8.22. *CMC surfaces in* 3*-dimensional space-forms are the constrained Willmore surfaces in* 3*-space admitting a* q*-conserved quantity, for some multiplier* q*.*

Next we establish a conserved quantity with respect to a general multiplier to a CMC surface in a 3-dimensional space-form. For each $t \in \mathbb{R}$, set

$$q_\infty^t := q_\infty + t * \eta_\infty.$$

Combined, Theorems 8.16 and 8.18 establish, in particular, the set of multipliers to Λ as the family q_∞^t, with $t \in \mathbb{R}$. In generalization of Proposition 8.20, we have:

Proposition 8.23. Λ *admits*

$$p_\infty^t(\lambda) := \lambda^{-1}\frac{1}{2}\,(H_\infty - it)N + v_\infty^T + \lambda\frac{1}{2}\,(H_\infty + it)N$$

as a q^t_∞-conserved quantity.

Proof. First of all, note that $d(v^T_\infty + \frac{1}{2}((H_\infty - it) + (H_\infty + it))N) = dv_\infty = 0$. The fact that $H_\infty = (N, v^\perp_\infty)$ is constant establishes

$$0 = d(N, v^\perp_\infty) = d(N, v_\infty) = (dN, v_\infty). \tag{8.26}$$

By (8.14), it follows that $(dN, v^T_\infty) = 0$ and, consequently, $\eta^{1,0}_\infty v^T_\infty = -\frac{1}{2}\, d^{1,0}N$. By (8.25), and, yet again, (8.14), we conclude then that

$$(q^t_\infty)^{1,0} v^T_\infty = q^{1,0}_\infty v^T_\infty - it\eta^{1,0}_\infty v^T_\infty = -\frac{1}{2}\mathcal{N}^{1,0} v^\perp_\infty + \frac{it}{2}\, d^{1,0}N = -\frac{1}{2}\mathcal{N}^{1,0}(H_\infty - it)N.$$

On the other hand, (8.14) establishes $\mathcal{D}^{0,1}N = 0$, and then $\mathcal{D}^{0,1}(H_\infty - it)N = 0$, in view of the constancy of H_∞. The conclusion follows, according to Theorem 7.2. □

We may, alternatively, use $q^{N,t}_\infty$ and $p^{N,t}_\infty$ to denote q^t_∞ and p^t_∞, respectively, in order to avoid the ambiguity with respect to the sign of N in the particular case $H_\infty = 0$.

We complete this section by remarking on the close relationship between the multiplier q_∞ and the Hopf differential. Fix a holomorphic chart z of M. For simplicity, write q^z_∞ for $(q_\infty)^z$, alternatively. Note that

$$(q_\infty)_{\delta_z}(\sigma_\infty)_z = \frac{1}{2}\,(\sigma_\infty \wedge dv^\perp_\infty)_{\delta_z}(\sigma_\infty)_z = -\frac{1}{2}\,(d_{\delta_z} v^\perp_\infty, (\sigma_\infty)_z)\sigma_\infty$$

and, therefore, by equation (8.23),

$$-2(q_\infty)_{\delta_z}(\sigma_\infty)_z = -(v^\perp_\infty, \mathcal{N}_{\delta_z}(\sigma_\infty)_z)\sigma_\infty = -(v^\perp_\infty, ((\sigma_\infty)_{zz})^\perp)\sigma_\infty.$$

We conclude that

$$q^z_\infty = -(v^\perp_\infty, ((\sigma_\infty)_{zz})^\perp).$$

Observe that, if $v^\perp_\infty \neq 0$, in which case $S^\perp = \langle v^\perp_\infty \rangle$, then, in view of the non-degeneracy of $S^\perp$ and according to equation (8.6), q^z_∞ is real if and only if so is k^z. Note that

$$q^z_\infty = -\lambda^{-1}(v^\perp_\infty, k^z),$$

for $\lambda \in \Gamma(\underline{\mathbb{R}})$ as in equation (8.6). For later reference, note that, in the case $v^\perp_\infty$ is non-zero,

$$(q^t_\infty)^z = (1 - itH^{-1}_\infty)(q_\infty)^z. \tag{8.27}$$

8.2.2. CMC surfaces in 3-space: an equation on the Hopf differential and the Schwarzian derivative. Constant mean curvature surfaces in 3-dimensional space-forms are, in particular, isothermic constrained Willmore surfaces. In view of the characterization of isothermic surfaces in space-forms in terms of the reality of the Hopf differential, the close relationship between the Hopf differential and the set of multipliers to a CMC surface in 3-space leads to a characterization of these surfaces in terms of the Schwarzian derivative and the Hopf differential, following the characterization of

constrained Willmore surfaces in space-forms presented in Section 5.4.

Let z be a holomorphic chart of M, which, in the case Λ has constant mean curvature in some space-form, we can choose so that k^z is real, cf. Lemma 8.7. If Λ is minimal in S_{v_∞}, then Λ is Willmore, so that, according to Lemma 5.11, $\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{\overline{c^z}}{2}\,k^z=0$ and, therefore,

$$\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{\overline{c^z}}{2}\,k^z=H_\infty k^z. \tag{8.28}$$

On the other hand, if Λ has non-zero constant mean curvature in S_{v_∞} and k^z is real, then, according to what was observed in Section 8.2.1, together with Lemma 5.11, q^z_∞ is a non-zero real-valued holomorphic function and, therefore, constant. We conclude that, if Λ has constant mean curvature $H_\infty\neq 0$ in S_{v_∞}, then we can choose z such that k^z is real and $q^z_\infty=H_\infty$ and, therefore, according to Lemma 5.11 equation (8.28) still holds. Furthermore, cf. [**14**]:

Lemma 8.24. *Λ has constant mean curvature $H\in\mathbb{R}$ in some space-form if and only if around each point there exists a holomorphic chart z of $(M,\mathcal{C}_\Lambda)$ such that k^z is real and*

$$\nabla^{S^\perp}_{\delta_{\bar z}}\nabla^{S^\perp}_{\delta_{\bar z}}k^z+\frac{\overline{c^z}}{2}k^z=Hk^z.$$

In that case, Λ is a q-constrained Willmore surface for q the quadratic differential defined locally by $q^z dz^2$ for $q^z:=H$, under the correspondence given by (5.14).

8.2.3. Spectral deformations of CMC surfaces in 3-space. As a class of constrained Willmore surfaces admitting a conserved quantity $p(\lambda)$ with $p(1)$ with non-zero orthogonal projection onto the central sphere congruence, the class of CMC surfaces in 3-space is known to be preserved by constrained Willmore spectral deformation. In this section, we investigate how the space-form and the mean curvature change under this deformation. The isothermic spectral deformation is known to preserve the constancy of the mean curvature of a surface in some space-form, cf. [**14**]. We establish it in our setting, along with verifying how the space-form and the mean curvature change under this deformation. We present the classical CMC spectral deformation by means of the action of a loop of flat metric connections on the class of CMC surfaces in 3-space (preserving the space-form and the mean curvature). We verify that all these deformations of CMC surfaces are closely related. We observe, in particular, that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation and that, in the particular case of minimal surfaces, the classical CMC spectral deformation coincides, up to reparametrization, with the constrained Willmore spectral deformation corresponding to the zero multiplier.

Suppose Λ has constant mean curvature H_∞ in S_{v_∞}. In that case, (Λ, η_∞) is an isothermic q^t_∞-constrained Willmore surface admitting $p^t_\infty(\lambda)$ as a q^t_∞-conserved quantity, for each $t \in \mathbb{R}$.

The constrained Willmore spectral deformation. For each $t \in \mathbb{R}$ and $\lambda \in S^1$, the constrained Willmore spectral deformation of Λ of parameter λ corresponding to the multiplier q^t_∞ is known to have constant mean curvature in some space-form, cf. Theorem 7.6, in view of (8.13). Next we investigate how the space-form and the mean curvature change with this deformation.

Fix $t \in \mathbb{R}$, $\lambda \in S^1$ and $\phi^\lambda_{q^t_\infty} : (\underline{\mathbb{R}}^{4,1}, d^\lambda_{q^t_\infty}) \to (\underline{\mathbb{R}}^{4,1}, d)$ an isomorphism. Set

$$v^\lambda_{q^t_\infty} := \phi^\lambda_{q^t_\infty}(v^T_\infty + ((\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))N).^9$$

Lemma 8.25. *$v^\lambda_{q^t_\infty}$ is a non-zero constant section of $\underline{\mathbb{R}}^{4,1}$.*

Proof. According to (8.13), v^T_∞ is non-zero and so is then $v^\lambda_{q^t_\infty}$. The fact that λ is unit establishes the reality of $\frac{it}{2}(\lambda - \lambda^{-1})$ and, consequently, that of $v^\lambda_{q^t_\infty} \in \Gamma((\underline{\mathbb{R}}^{4,1})^{\mathbb{C}})$. In its turn, the fact that p^t_∞ is a q^t_∞-conserved quantity of Λ ensures the constancy of the section $\phi^\lambda_{q^t_\infty}(p^t_\infty(\lambda)) = v^\lambda_{q^t_\infty}$ of $\underline{\mathbb{R}}^{4,1}$: $d(\phi^\lambda_{q^t_\infty}(p^t_\infty(\lambda))) = \phi^\lambda_{q^t_\infty}(d^\lambda_{q^t_\infty} p^t_\infty(\lambda)) = 0$. $\square$

In view of the fact that $p^t_\infty(\lambda)$ is a q^t_∞-conserved quantity of Λ, together with (8.13), Theorem 7.6 establishes the constrained Willmore spectral deformation of Λ, of parameter λ, corresponding to the multiplier q^t_∞ as a constrained Willmore surface (in 3-dimensional space-form) admitting a conserved quantity, or, equivalently, as a CMC surface in some 3-space. Specifically, according to Proposition 8.21:

Theorem 8.26. *The spectral deformation of Λ of parameter λ corresponding to the multiplier q^t_∞ is a CMC surface in $S_{v^\lambda_{q^t_\infty}}$.*

Recall that the constrained Willmore spectral deformation preserves the central sphere congruence, as observed in (6.14), to conclude that:

Proposition 8.27. *Let $\Lambda^\lambda_{q^t_\infty}$ be the spectral deformation of Λ, of parameter λ, corresponding to the multiplier q^t_∞. The mean curvature $H^\lambda_{q^t_\infty}$ of $\Lambda^\lambda_{q^t_\infty}$ in $S_{v^\lambda_{q^t_\infty}}$ relates to the mean curvature of Λ in S_{v_∞} by*

$$H^\lambda_{q^t_\infty} = \mid \mathrm{Re}\,(\lambda H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))\mid .$$

Proof. The mean curvature of $\phi^\lambda_{q^t_\infty}\Lambda$ in $S_{v^\lambda_{q^t_\infty}}$ is given by $H^\lambda_{q^t_\infty} = (\pi^\perp_{\lambda,t}\, v^\lambda_{q^t_\infty}, \pi^\perp_{\lambda,t}\, v^\lambda_{q^t_\infty})^{\frac{1}{2}}$, for $\pi^\perp_{\lambda,t}$ the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto the normal bundle to $S_{\phi^\lambda_{q^t_\infty}\Lambda} = \phi^\lambda_{q^t_\infty} S$,

[9] Recall that $v^\perp_\infty = H_\infty N$, so that, in the particular case $t = 0$, $v^\lambda_{q^t_\infty} = \phi^\lambda_{q^t_\infty}(v^T_\infty + (\mathrm{Re}\,\lambda)v^\perp_\infty)$.

and, therefore,

$$\begin{aligned} H^{\lambda}_{q^t_\infty} &= (((\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))N, ((\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}))N)^{\frac{1}{2}} \\ &= \mid (\mathrm{Re}\lambda)H_\infty + \frac{it}{2}(\lambda - \lambda^{-1}) \mid \\ &= \mid \mathrm{Re}\,(\lambda H_\infty + \frac{it}{2}(\lambda - \lambda^{-1})) \mid, \end{aligned}$$

having in consideration that, as Λ is unit, $\frac{it}{2}(\lambda - \lambda^{-1})$ is real. $\square$

It is interesting to remark that, for $t = 0$, the deformations corresponding to the parameters i and $-i$ are minimal surfaces and, therefore, Willmore surfaces, even when Λ is not a Willmore surface.

The isothermic spectral deformation. Cf. [**14**], the isothermic spectral deformation preserves the class of CMC surfaces in 3-dimensional space-forms. Next we establish it in our setting and investigate how the space-form and the mean curvature change with this deformation.

First of all, note that, if $H_\infty \neq 0$, then $\eta_\infty = H_\infty^{-1} q_\infty$, and, therefore, the fact that q_∞ is a multiplier to Λ ensures that

$$d^{\mathcal{D}} * \mathcal{N} \neq 2[\eta_\infty \wedge *\mathcal{N}],$$

η_∞ is not a multiplier to Λ. If, on the other hand, Λ is minimal in S_{v_∞}, then the set of multipliers to Λ is the vector space $\langle *\eta_\infty \rangle_{\mathbb{R}} \neq 0$, in which case we conclude, yet again, that η_∞ is not a multiplier to Λ. There is, therefore, no risk of ambiguity on the notation $d^t_{\eta_\infty}$.

Fix $t \in \mathbb{R}$ and $\phi^t_{\eta_\infty} : (\underline{\mathbb{R}}^{4,1}, d^t_{\eta_\infty}) \to (\underline{\mathbb{R}}^{4,1}, d)$ an isomorphism. Set

$$v^t_{\eta_\infty} := \phi^t_{\eta_\infty}(v_\infty + \frac{t}{2}\, N).$$

Lemma 8.28. $v^t_{\eta_\infty}$ *is a non-zero constant section of* $\underline{\mathbb{R}}^{4,1}$.

Proof. The orthogonal projection of $(\phi^t_{\eta_\infty})^{-1} v^t_{\eta_\infty}$ onto S is v^T_∞, which is non-zero, cf. (8.13). Thus $v^t_{\eta_\infty}$ is non-zero. On the other hand, the constancy of v_∞, together with the fact that η_∞ vanishes on $S^\perp$, gives

$$\begin{aligned} dv^t_{\eta_\infty} &= \phi^t_{\eta_\infty}(d^t_{\eta_\infty} v^t_{\eta_\infty}) \\ &= \phi^t_{\eta_\infty}(dv_\infty + \frac{t}{2}\, dN + t\eta_\infty v_\infty + \frac{t^2}{2}\, \eta_\infty N) \\ &= \phi^t_{\eta_\infty}(\frac{t}{2}\, (dN, v_\infty)\sigma_\infty). \end{aligned}$$

The constancy of H_∞, and, in particular, (8.26), establishes then the constancy of $v^t_{\eta_\infty}$, completing the proof. $\square$

The isothermic spectral deformation provides, in particular, a deformation of CMC surfaces in 3-space. In fact:

Theorem 8.29. *The isothermic (t, η_∞)-transformation of Λ is a CMC surface in $S_{v^t_{\eta_\infty}}$.*

PROOF. The mean curvature $H^t_{\eta_\infty}$ of $\phi^t_{\eta_\infty}\Lambda$ in the space-form $S_{v^t_{\eta_\infty}}$ is given by $H^t_{\eta_\infty} = (\pi^\perp_t v^t_{\eta_\infty}, \pi^\perp_t v^t_{\eta_\infty})^{\frac{1}{2}}$, for $\pi^\perp_t$ the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto the normal bundle to $S_{\phi^t_{\eta_\infty}\Lambda} = \phi^t_{\eta_\infty} S$, and, therefore,

$$(H^t_{\eta_\infty})^2 = (v^\perp_\infty + \frac{t}{2}\,N, v^\perp_\infty + \frac{t}{2}\,N) = H^2_\infty + t(v^\perp_\infty, N) + \frac{t^2}{4} = (H_\infty + \frac{t}{2})^2.$$

□

In the proof of Theorem 8.29, we have verified, in particular, that:

Proposition 8.30. *Let $\Lambda^t_{\eta_\infty}$ be the isothermic (t, η_∞)-transformation of Λ. The mean curvature $H^t_{\eta_\infty}$ of $\Lambda^t_{\eta_\infty}$ in $S_{v^t_{\eta_\infty}}$ relates to the mean curvature of Λ in S_{v_∞} by*

$$(H^t_{\eta_\infty})^2 = (H_\infty + \frac{t}{2})^2.$$

Remark 8.31. *Let k^t_∞ denote the curvature of $S_{v^t_\infty}$. According to* (8.9)*, the family $\phi^t_{\eta_\infty}\sigma_\infty$, on $t \in \mathbb{R}$, constitutes an isothermic deformation of σ_∞ with*

$$k^t_{\eta_\infty} + (H^t_{\eta_\infty})^2 = -(v^t_{\eta_\infty}, v^t_{\eta_\infty}) + (v^\perp_\infty + \frac{t}{2}N, v^\perp_\infty + \frac{t}{2}N) = -(v^T_\infty, v^T_\infty),$$

independent of t.

The classical CMC spectral deformation. The isometric deformation of surfaces in $\mathbb{R}^3$ preserving the mean curvature, or, equivalently,[10] both principal curvatures, was first studied by O. Bonnet. Bonnet [**8**] (see also, for example, [**5**] and [**19**]) proved that a CMC surface in Euclidean 3-space admits a (non-trivial) one-parameter family of isometrical deformations preserving both principal curvatures. In [**14**], F. Burstall et al. present an action of $\mathbb{C}\backslash\{0\}$ on the class of constant mean curvature surfaces in 3-space, in terms of the Hopf differential and the Schwarzian derivative. The particular action of S^1 preserves the metric, the space-form and the mean curvature - this is the classical CMC spectral deformation. In this section, we present the classical CMC spectral deformation by means of the action of a loop of flat metric connections on the class of CMC surfaces in 3-space.

[10]Cf. Gauss's *theorema egregium*, isometric deformation of surfaces in Euclidean 3-space preserves the Gaussian curvature, which, in view of (2.3) and (2.4), makes clear that the mean curvature is preserved by such a deformation if and only if so are both principal curvatures.

For each $\lambda \in S^1$, set

$$d^\lambda_\infty := \mathcal{D} + \lambda\mathcal{N}^{1,0} + \lambda^{-1}\mathcal{N}^{0,1} + 2(\lambda - 1)q^{1,0}_\infty + 2(\lambda^{-1} - 1)q^{0,1}_\infty,$$

defining a real connection on $(\underline{\mathbb{R}}^{4,1})^{\mathbb{C}}$. As $\mathcal{D}$ is metric, and both $\mathcal{N}$ and q_∞ are skew-symmetric, d^λ_∞ is metric too, for each λ.

Theorem 8.32. *d^λ_∞ is a flat connection, for each $\lambda \in S^1$.*

Proof. The curvature tensor of d^λ_∞ is given by

$$\begin{aligned} R^{d^\lambda_\infty} &= (\lambda - \lambda^{-1})d^{\mathcal{D}}\mathcal{N}^{1,0} + 2(\lambda - 1)d^{\mathcal{D}}q^{1,0}_\infty + 2(\lambda^{-1} - 1)d^{\mathcal{D}}q^{0,1}_\infty \\ &\quad + (2 - 2\lambda)\,[\mathcal{N}^{1,0} \wedge q^{0,1}_\infty] + (2 - 2\lambda^{-1})\,[\mathcal{N}^{0,1} \wedge q^{1,0}_\infty] - (\lambda^{-1} + \lambda - 2)[q^{1,0}_\infty \wedge q^{0,1}_\infty], \end{aligned}$$

according to Codazzi and Gauss-Ricci equations. The fact that Λ is a CMC surface in S_{v_∞}, and, therefore, a q_∞-constrained Willmore surface, establishes, in particular, $d^{\mathcal{D}}q^{1,0}_\infty = 0 = d^{\mathcal{D}}q^{0,1}_\infty$ and, cf. (**??**), $[q^{1,0}_\infty \wedge q^{0,1}_\infty] = 0$. The fact that Λ is a CMC surface in S_{v_∞} establishes, on the other hand, (Λ, η_∞) as isothermic, and, therefore, $[\mathcal{N} \wedge \eta_\infty] = 0$, cf. Lemma 8.4. Hence $R^{d^\lambda_\infty} = 0$ if and only if either $\lambda = \pm 1$ or $d^{\mathcal{D}}\mathcal{N}^{1,0} = 2[\mathcal{N}^{1,0} \wedge q^{0,1}_\infty]$. The fact that Λ is a q_∞-constrained Willmore surface gives, on the other hand, $d^{\mathcal{D}}(-i\mathcal{N}^{1,0} + i\mathcal{N}^{0,1}) = -2i[q_\infty \wedge \mathcal{N}^{1,0}] + 2i[q_\infty \wedge \mathcal{N}^{0,1}]$, which completes the proof. $\square$

Fix $\lambda \in S^1$ and $\phi^\lambda_\infty : (\underline{\mathbb{R}}^{4,1}, d^\lambda_\infty) \to (\underline{\mathbb{R}}^{4,1}, d)$ an isomorphism. Set

$$v^\lambda_\infty := \phi^\lambda_\infty v_\infty.$$

Lemma 8.33. *v^λ_∞ is a non-zero constant section of $\underline{\mathbb{R}}^{4,1}$.*

Proof. According to equations (8.22) and (8.23), together with the fact that q_∞ vanishes on $S^\perp$,

$$d^\lambda_\infty v_\infty = \mathcal{D}v^T_\infty + \lambda\mathcal{N}^{1,0}v^\perp_\infty + \lambda^{-1}\mathcal{N}^{0,1}v^\perp_\infty + 2(\lambda - 1)q^{1,0}_\infty v^T_\infty + 2(\lambda^{-1} - 1)q^{0,1}_\infty v^T_\infty,$$

and, consequently, by (8.25), followed by (8.24), $d^\lambda_\infty v_\infty = \mathcal{D}v^T_\infty + \mathcal{N}v^\perp_\infty = 0$. We conclude that v^λ_∞ is constant: $dv^\lambda_\infty = d(\phi^\lambda_\infty v_\infty) = \phi^\lambda_\infty(d^\lambda_\infty v_\infty) = 0$. Inequation (8.13) establishes v^λ_∞ as non-zero. $\square$

As $\mathcal{N}\Lambda = 0 = q_\infty\Lambda$, given $\sigma \in \Gamma(\Lambda)$,

$$d^\lambda_\infty \sigma = d\sigma, \tag{8.29}$$

showing that Λ is still a d^λ_∞-surface, or, equivalently, that the transformation Λ^λ_∞ of Λ, defined by the flat metric connection d^λ_∞, is still a surface. Furthermore:

Theorem 8.34. *The transformation Λ^λ_∞ of Λ defined by the flat metric connection d^λ_∞ is a CMC surface in*

$$S_{v^\lambda_\infty} = \phi^\lambda_\infty S_{v_\infty}.$$

Before proceeding to the proof of the theorem, observe that Λ and Λ^λ_∞ share the central sphere congruence. For that, first note that, according to equation (8.29), given $\sigma \in \Gamma(\Lambda)$ never-zero,

$$g_{\phi^\lambda_\infty \sigma} = g^{d^\lambda_\infty}_\sigma = g_\sigma, \tag{8.30}$$

establishing

$$\mathcal{C}_{\Lambda^\lambda_\infty} = \mathcal{C}_\Lambda.$$

Yet again, in view of equation (8.29), it follows that, given a holomorphic chart z of $(M, \mathcal{C}^{d^\lambda_\infty}_\Lambda) = (M, \mathcal{C}_\Lambda)$, $(d^\lambda_\infty)_{\delta_{\bar{z}}}(d^\lambda_\infty)_{\delta_z}\sigma = (d^\lambda_\infty)_{\delta_{\bar{z}}}\sigma_z = \sigma_{z\bar{z}} + \lambda^{-1}\pi_{S^\perp}\sigma_{z\bar{z}} + 2(\lambda^{-1}-1)q^{0,1}_\infty \sigma_z$ and, therefore, $(d^\lambda_\infty)_{\delta_{\bar{z}}}(d^\lambda_\infty)_{\delta_z}\sigma = \sigma_{z\bar{z}}$, as $q^{0,1}_\infty \in \Omega^1(\Lambda \wedge \Lambda^{1,0})$. We conclude that $S^{d^\lambda_\infty} = S$ and, ultimately, according to (3.3), that

$$S_{\phi^\lambda_\infty \Lambda} = \phi^\lambda_\infty S. \tag{8.31}$$

Now we proceed to the proof of Theorem 8.34.

Proof. According to (8.31), the mean curvature H^λ_∞ of $\phi^\lambda_\infty \Lambda$ in $S_{v^\lambda_\infty}$ is given by $H^\lambda_\infty = (\pi^\perp_\lambda v^\lambda_\infty, \pi^\perp_\lambda v^\lambda_\infty)^{\frac{1}{2}}$, for $\pi^\perp_\lambda$ the orthogonal projection of $\underline{\mathbb{R}}^{4,1}$ onto $\phi^\lambda_\infty S^\perp$, and, therefore, $H^\lambda_\infty = (v^\perp_\infty, v^\perp_\infty)^{\frac{1}{2}} = H_\infty$. $\square$

In the proof of Theorem 8.34, we have, in particular, verified that:

Proposition 8.35. *The mean curvature H^λ_∞ of $\Lambda^\lambda_{q_\infty}$ in $S_{v^\lambda_\infty}$ coincides with the mean curvature of Λ in S_{v_∞},*

$$H^\lambda_\infty = H_\infty.$$

The loop of flat metric connections d^λ_∞ defines a conformal S^1-deformation of Λ into CMC surfaces in a fixed space-form, preserving the mean curvature.

Remark 8.36. *The family $\phi^\lambda_\infty \sigma_\infty$, with $\lambda \in S^1$, is a family of isometrical deformations of σ_∞ in a fixed space-form, preserving the mean curvature.*

We complete this section by verifying that the deformation defined by the loop of flat metric connections d^λ_∞ is the classical CMC spectral deformation, described in [**14**] in terms of the Hopf differential and the Schwarzian derivative. Fix a holomorphic chart z of $(M, \mathcal{C}_{\Lambda^\lambda_\infty}) = (M, \mathcal{C}_\Lambda)$. According to (8.30), $g_{\phi^\lambda_\infty \sigma^z} = g_z$, showing that $\phi^\lambda_\infty \sigma^z$ is the normalized section of $\phi^\lambda_\infty \Lambda$ with respect to z. For simplicity, write q^z_∞ for $(q_\infty)^z$. In view of (8.29),

$$\begin{aligned}
(\phi^\lambda_\infty \sigma^z)_{zz} &= (\phi^\lambda_\infty \sigma^z_z)_z \\
&= \phi^\lambda_\infty((d^\lambda_\infty)_{\delta_z}\sigma^z_z) \\
&= \phi^\lambda_\infty(\mathcal{D}_{\delta_z}\sigma^z_z + \lambda \mathcal{N}_{\delta_z}\sigma^z_z + 2(\lambda-1)(q_\infty)_{\delta_z}\sigma^z_z) \\
&= \phi^\lambda_\infty(\pi_S \sigma^z_{zz} + \lambda \pi_{S^\perp}\sigma^z_{zz} - (\lambda-1)\, q^z_\infty \sigma^z)
\end{aligned}$$

and, ultimately,

$$(\phi_\infty^\lambda \sigma^z)_{zz} = -\frac{1}{2}\left(c^z + 2(\lambda - 1)\, q_\infty^z\right) \phi_\infty^\lambda \sigma^z + \lambda \phi_\infty^\lambda k^z.$$

We conclude that $(k_\infty^\lambda)^z$ and $(c_\infty^\lambda)^z$, the Hopf differential and the Schwarzian derivative, respectively, of $\phi_\infty^\lambda \Lambda$ with respect to z, relate to those of Λ by

$$(k_\infty^\lambda)^z = \lambda \phi_\infty^\lambda k^z, \quad (c_\infty^\lambda)^z = c^z + 2(\lambda - 1)\, q_\infty^z.$$

By Lemma 5.5, the conclusion follows.

Isothermic vs. constrained Willmore vs. classical CMC spectral deformations. How are the constrained Willmore, isothermic and classical CMC spectral deformations of a CMC surface in 3-space related? In this section, we compare the families of flat metric connections that define each of these deformations. We observe, in particular, that the classical CMC spectral deformation can be obtained as composition of isothermic and constrained Willmore spectral deformation and that, in the particular case of minimal surfaces, the classical CMC spectral deformation coincides, up to reparametrization, with the constrained Willmore spectral deformation corresponding to the zero multiplier.

We start by introducing some terminology. Note that, according to (5.2), given z and ω holomorphic charts of $(M, \mathcal{C}_\Lambda)$, k^z vanishes if and only if k^ω does. We refer to the points where the Hopf differential of Λ vanishes as *umbilic points* of Λ, in coherence with the classical notion of umbilic point of a surface in Euclidean 3-space, cf. Proposition A.1 below, in Appendix A (having in consideration that Λ is isothermic).

Fix z a holomorphic chart of M and let σ^z be the normalized section of Λ with respect to z. Given $t, t' \in \mathbb{R}$ and $\lambda \in S^1$, $d^\lambda_{q^t_\infty} = d^{t'}_{\eta_\infty}$ forces, in particular, $d^\lambda_{q^t_\infty} \sigma^z_z = d^{t'}_{\eta_\infty} \sigma^z_z$, or, equivalently,

$$\mathcal{D}_{\delta_z} \sigma^z_z + \lambda^{-1} \mathcal{N}_{\delta_z} \sigma^z_z + (\lambda^{-2} - 1)(q^t_\infty)_{\delta_z} \sigma^z_z = \mathcal{D}_{\delta_z} \sigma^z_z + \mathcal{N}_{\delta_z} \sigma^z_z + t'(\eta_\infty)_{\delta_z} \sigma^z_z,$$

or, yet again, equivalently,

$$(\lambda^{-1} - 1) k^z = 0, \quad (\lambda^{-2} - 1)(q^t_\infty)_{\delta_z} \sigma^z_z = t'(\eta_\infty)_{\delta_z} \sigma^z_z,$$

in view of the fact that $\operatorname{Im} q^t_\infty, \operatorname{Im} \eta_\infty \subset S$. By (8.16) and according to Remark 5.8, $(\eta_\infty)_{\delta_z} \sigma^z_z \neq 0$. We conclude that, away from umbilics, $d^\lambda_{q^t_\infty} = d^{t'}_{\eta_\infty}$ holds if and only if $\lambda = 1$ and $t' = 0$, in which case,

$$d^1_{q^t_\infty} = d = d^0_{\eta_\infty},$$

for all $t \in \mathbb{R}$. Similarly, we conclude that, away from umbilics, given $\lambda, \lambda' \in S^1$ and $t, t' \in \mathbb{R}$, $d^\lambda_{q^t_\infty} = d^{\lambda'}_{q^{t'}_\infty}$ if and only if $\lambda = \lambda'$ and either $\lambda = \pm 1$ or $t = t'$. Lastly, if

$d^\lambda_\infty = d^{\lambda'}_{q^t_\infty}$, for some $\lambda, \lambda' \in S^1$ and $t \in \mathbb{R}$, then

$$2(\lambda - 1)(q_\infty)_{\delta_z}\sigma^z_z = ((\lambda')^{-2} - 1)(H_\infty - it)(\eta_\infty)_{\delta_z}\sigma^z_z$$

and either $\lambda' = \lambda^{-1}$ or $k^z = 0$; and, therefore, away from umbilics, $\lambda' = \lambda^{-1}$ and either $\lambda = 1$ or

$$(2H_\infty - (\lambda + 1)(H_\infty - it))(\eta_\infty)_{\delta_z}\sigma^z_z = 0. \tag{8.32}$$

In its turn, unless $\lambda = -1$, equation (8.32) forces t to be

$$t_\lambda := iH_\infty \frac{1-\lambda}{1+\lambda} \in \mathbb{R}.$$

Conversely, one verifies that, for all $\lambda \neq \pm 1$,

$$d^\lambda_\infty = d^{\lambda^{-1}}_{q^{t_\lambda}_\infty},$$

which, in fact, extends to all $\lambda \neq -1$: $d^1_\infty = d^1_{q^0_\infty}$. One verifies that, given $t \in \mathbb{R}$, we have $d^{-1}_\infty = d^{-1}_{q^t_\infty}$ if and only if $H_\infty = 0$. Note, in particular, that, if $H_\infty = 0$, then $d^\lambda_\infty = d^{\lambda^{-1}}_0$, for all $\lambda \in S^1$.

For each $t \in \mathbb{R}$, we have a multiplier q^t_∞ and, therefore, a loop of q^t_∞-constrained Willmore spectral deformation parameters $\lambda \in S^1$. The set of constrained Willmore spectral deformation parameters is described, in this way, as the cylinder of points (t, λ) with $t \in \mathbb{R}$ and $\lambda \in S^1$. A transformation corresponding to a parameter in the line $(t, 1)$, with $t \in \mathbb{R}$, is trivial, and so is the transformation corresponding to the parameter 0 in the real line of isothermic spectral deformation parameters. The transformations corresponding to parameters in the line $(t, -1)$ do not depend on $t \in \mathbb{R}$. For minimal surfaces, the classical CMC spectral deformation coincides, up to reparametrization, with the constrained Willmore spectral deformation corresponding to the zero multiplier. Furthermore, for a general surface, the classical CMC spectral deformation of parameter other than -1 can be obtained as constrained Willmore spectral deformation of parameters in the curve $(t_\lambda, \lambda^{-1})$, with $\lambda \in S^1$. As for the classical CMC deformation of parameter -1 of non-minimal surfaces, it is not clear that it can be obtained by constrained Willmore spectral deformation alone. However, such transformations can be obtained as composition of isothermic and constrained Willmore spectral deformation, as we verify next. Given $\lambda \in S^1$,

$$\begin{aligned} d^{\lambda^{-1}}_\infty &= d^\lambda_{q_\infty} + (-\lambda^{-2} + 2\lambda^{-1} - 1)\, q^{1,0}_\infty + (-\lambda^2 + 2\lambda - 1)\, q^{0,1}_\infty \\ &= d^\lambda_{q_\infty} + H_\infty(2 - \lambda - \lambda^{-1})(\lambda^{-1}\eta^{1,0}_\infty + \lambda\, \eta^{0,1}_\infty) \end{aligned}$$

and, ultimately,

$$d^{\lambda^{-1}}_\infty = d^\lambda_{q_\infty} + 2H_\infty(1 - \operatorname{Re}\lambda)\, \eta^\lambda_\infty, \tag{8.33}$$

for $\eta^\lambda_\infty := (\eta_\infty)_\lambda$ as in Section 8.1.5. Fix $\lambda \in S^1$ and $\phi^\lambda_{q_\infty} : (\underline{\mathbb{R}}^{4,1}, d^\lambda_{q_\infty}) \to (\underline{\mathbb{R}}^{4,1}, d)$ an isomorphism. Cf. Section 8.1.5, $(\phi^\lambda_{q_\infty}\Lambda, \hat{\eta}^\lambda_\infty)$ is isothermic, for

$$\hat{\eta}^\lambda_\infty := \mathrm{Ad}_{\phi^\lambda_{q_\infty}} \eta^\lambda_\infty.$$

Set

$$r_\lambda := 2H_\infty(1 - \mathrm{Re}\,\lambda)$$

and fix an isomorphism $\phi^r_\lambda : (\underline{\mathbb{R}}^{4,1}, d + r_\lambda \hat{\eta}^\lambda_\infty) \to (\underline{\mathbb{R}}^{4,1}, d)$. Following equation (8.33), we get that

$$\phi^r_\lambda \phi^\lambda_{q_\infty} \circ d^{\lambda^{-1}}_\infty = \phi^r_\lambda \circ (d \circ \phi^\lambda_{q_\infty} + r_\lambda \hat{\eta}^\lambda_\infty \phi^\lambda_{q_\infty}) = d \circ \phi^r_\lambda \phi^\lambda_{q_\infty},$$

the isometry

$$\phi^r_\lambda \phi^\lambda_{q_\infty} : (\underline{\mathbb{R}}^{4,1}, d^{\lambda^{-1}}_\infty) \to (\underline{\mathbb{R}}^{4,1}, d)$$

preserves connections. We conclude that the isothermic $(r_\lambda, \hat{\eta}^\lambda_\infty)$-transformation of the constrained Willmore spectral deformation of parameter λ of Λ corresponding to the multiplier q_∞ coincides with the classical CMC spectral deformation of parameter λ^{-1} of Λ,

$$\Lambda^{\lambda^{-1}}_\infty = (\Lambda^\lambda_{q_\infty})^{r_\lambda}_{\hat{\eta}^\lambda_\infty}.$$

8.2.4. Constrained Willmore Bäcklund transformation of CMC surfaces in 3-space. Characterized as the class of constrained Willmore surfaces in 3-dimensional space-forms admitting a conserved quantity, the class of CMC surfaces in 3-space is known to be preserved by constrained Willmore Bäcklund transformation, for special choices of parameters. In this section, we verify that both the space-form and the mean curvature are preserved under this transformation. We remark on the fact that constrained Willmore Bäcklund transformation of non-minimal CMC surfaces in 3-space preserves conformal curvature line coordinates.

Suppose Λ has constant mean curvature H_∞ in S_{v_∞}. Then, for each $t \in \mathbb{R}$, Λ is a q^t_∞-constrained Willmore surface admitting $p^t_\infty(\lambda)$ as a q^t_∞-conserved quantity. Fix $t \in \mathbb{R}$. Suppose α_t, L^{α_t} are constrained Willmore Bäcklund transformation parameters to Λ, corresponding to the multiplier q^t_∞, satisfying the condition

$$p^t_\infty(\alpha_t) \perp L^{\alpha_t}, \tag{8.34}$$

and let Λ^{*_t} be the constrained Willmore Bäcklund transform of Λ of parameters α_t, L^{α_t}. Suppose Λ^{*_t} immerses. Let r^{*_t} denote $r^{\alpha_t}_{L^{\alpha_t}}$. According to Theorem 7.7, Λ^{*_t} is a $(q^t_\infty)^*$-constrained Willmore surface admitting

$$(p^t_\infty)^*(\lambda) = r^{*_t}(1)^{-1} r^{*_t}(\lambda) p^t_\infty(\lambda)$$

as a $(q^t_\infty)^*$-conserved quantity, and, therefore, a CMC surface in some space-form. Furthermore, according to Proposition 8.21:

Theorem 8.37. *Suppose Λ is a CMC surface in S_{v_∞}. Suppose α_t, L^{α_t} are constrained Willmore Bäcklund transformation parameters to Λ, corresponding to the multiplier q^t_∞, verifying condition* (8.34)*. Then the constrained Willmore Bäcklund transform of Λ of parameters α_t, L^{α_t} still is a CMC surface in S_{v_∞}, provided that it immerses.*

Next we investigate how the mean curvature changes under this transformation. Let S^{*_t} be the central sphere congruence of Λ^{*_t} and T_{*_t} and $\perp_{*_t}$ indicate the orthogonal projections of $\underline{\mathbb{R}}^{4,1}$ onto S^{*_t} and $(S^{*_t})^\perp$, respectively.

Proposition 8.38. *Set $N^{*_t} := -r^{*_t}(1)^{-1}N$. Then*

$$v_\infty^{\perp_{*_t}} = H_\infty N^{*_t} \tag{8.35}$$

and

$$(p^t_\infty)^*(\lambda) = \lambda^{-1}\frac{1}{2}\,(H_\infty - it)N^{*_t} + v_\infty^{T_{*_t}} + \lambda\frac{1}{2}\,(H_\infty + it)N^{*_t},$$

for all λ.

Before proceeding to the proof of the proposition:

Remark 8.39. *Let ρ denote reflection across S. According to* (6.17)*, $r^{*_t}(0)|_{S^\perp}$ and $r^{*_t}(\infty)|_{S^\perp}$ are orthogonal transformations of $S^\perp$ and, therefore, as $S^\perp$ is a* rank 1 *bundle, $r^{*_t}(0)|_{S^\perp}, r^{*_t}(\infty)|_{S^\perp} \in \{\pm I\}$. On the other hand,*

$$r^{*_t}(0)|_{S^\perp} = q_{\overline{\alpha_t}^{-1},\overline{L^{\alpha_t}}}(0)|_{S^\perp} = I\begin{cases} -1 & \text{on } S^\perp \cap (\overline{L^{\alpha_t}} \oplus \rho\overline{L^{\alpha_t}}) \\ 1 & \text{on } S^\perp \cap (\overline{L^{\alpha_t}} \oplus \rho\overline{L^{\alpha_t}})^\perp \end{cases},$$

so that, if $r^{*_t}(0)|_{S^\perp}$ were identity, then $S^\perp \cap (\overline{L^{\alpha_t}} \oplus \rho\overline{L^{\alpha_t}})^\perp$ would be $S^\perp$ and, therefore, $L^{\alpha_t} \subset S$, in which case, $\rho L^{\alpha_t} = L^{\alpha_t}$, which, as L^{α_t} is null, contradicts the fact that $\rho L^{\alpha_t} \cap (L^{\alpha_t})^\perp = \{0\}$. Hence $r^{*_t}(0)|_{S^\perp} = -I$. Similarly, the fact that $\tilde{L}^{\alpha_t}$ is null and $\rho\tilde{L}^{\alpha_t} \cap (\tilde{L}^{\alpha_t})^\perp = \{0\}$ establishes $r^{*_t}(\infty)|_{S^\perp} = p_{\alpha_t,\tilde{L}^{\alpha_t}}(\infty)|_{S^\perp} = -I$. Thus

$$r^{*_t}(0)|_{S^\perp} = -I = r^{*_t}(\infty)|_{S^\perp}. \tag{8.36}$$

Now we proceed to the proof of Proposition 8.38.

Proof. Write $(p^t_\infty)^*(\lambda) = \lambda^{-1}v_t + v^t_0 + \lambda\overline{v_t}$ with $v^t_0 \in \Gamma(S^{*_t})$ and $v_t \in \Gamma((S^{*_t})^\perp)$. The fact that $v_\infty = (p^t_\infty)^*(1) = v_t + v^t_0 + \overline{v_t}$ establishes $v^t_0 = v_\infty^{T_{*_t}}$ and

$$v_t + \overline{v_t} = v_\infty^{\perp_{*_t}}. \tag{8.37}$$

On the other hand,

$$\begin{aligned} v_t &= \lim_{\lambda\to 0}\lambda(\lambda^{-1}v_t + v^t_0 + \lambda\overline{v_t}) \\ &= \lim_{\lambda\to 0}\lambda(p^t_\infty)^*(\lambda) \\ &= r^{*_t}(1)^{-1}r^{*_t}(0)\lim_{\lambda\to 0}\lambda p^t_\infty(\lambda) \\ &= \frac{1}{2}\,r^{*_t}(1)^{-1}r^{*_t}(0)(H_\infty - it)N \end{aligned}$$

and, therefore, by (8.36),

$$v_t = \frac{1}{2}\,(H_\infty - it)N^{*_t}.$$

Next let $K^t \in \Gamma(O(\underline{\mathbb{C}}^{n+2}))$ be K, as defined in Proposition 6.27, for $\alpha = \alpha_t$ and $\beta = \overline{\alpha_t}^{-1}$. According to (6.39), K^t preserves $S^\perp$. Hence K^tN is a unit section of $S^\perp$, which equation (6.49) ensures to be real. Thus $K^tN = \pm N$. If $K^tN = -N$, then, according to equation (6.46), $\overline{N^{*_t}} = -r^{*_t}(1)^{-1}K^tN = -N^{*_t}$ and, therefore, iN^{*_t} is a real section of $(S^{*_t})^\perp$ with $(iN^{*_t}, iN^{*_t}) = -1$, which contradicts the fact that the metric in $(S^{*_t})^\perp$ has no signature. We conclude that $K^tN = N$. It follows that $\overline{N^{*_t}} = -r^{*_t}(1)^{-1}K^tN = N^{*_t}$, $\overline{N^{*_t}}$ is real, and, therefore, by (8.37), $v_\infty^{\perp_{*_t}} = H_\infty N^{*_t}$, completing the proof. □

Equation (8.35) establishes immediately that:

Proposition 8.40. *The mean curvature $H_\infty^{*_t}$ of Λ^{*_t} in S_{v_∞} relates to the mean curvature of Λ in S_{v_∞} by*

$$H_\infty^{*_t} = H_\infty.$$

Let $\eta_\infty^{N^{*_t}}$ be the *canonical* form establishing the isothermic condition of Λ^{*_t} as a CMC surface in S_{v_∞}, as established in Section 8.2.1, $\eta_\infty^{N^{*_t}} := \frac{1}{2}\,\sigma_\infty^{*_t} \wedge dN^{*_t}$, for $\sigma_\infty^{*_t}$ the surface in S_{v_∞} defined by Λ^{*_t}. Let $q_\infty^{*_t}$ be the *canonical* multiplier to Λ^{*_t} as a CMC surface in S_{v_∞}, i.e., $q_\infty^{*_t} := H_\infty \eta_\infty^{N^{*_t}}$. For each $t' \in \mathbb{R}$, let $(q_\infty^{*_t})^{t'}$ denote the multiplier $q_\infty^{*_t} + t' * \eta_\infty^{N^{*_t}}$ to Λ^{*_t} and $(p_\infty^{*_t})^{t'}$ be the *canonical* $(q_\infty^{*_t})^{t'}$-conserved quantity of Λ^{*_t}, as established in Proposition 8.23, $(p_\infty^{*_t})^{t'} := \lambda^{-1}\frac{1}{2}\,(H_\infty - it')N^{*_t} + v_\infty^{T_{*_t}} + \lambda\frac{1}{2}\,(H_\infty + it')N^{*_t}$. Then

$$(p_\infty^t)^* = (p_\infty^{*_t})^t$$

and, therefore, according to Remark 7.4,

$$(q_\infty^t)^* = (q_\infty^{*_t})^t.$$

By Proposition 6.36, it follows that the quadratic differentials $(q_\infty^t)_Q$ and $((q_\infty^{*_t})^t)_Q$ coincide, $(q_\infty^t)_Q = ((q_\infty^{*_t})^t)_Q$. Recall that Λ and Λ^{*_t} induce the same conformal structure in M. We conclude that, given z a holomorphic chart of $(M, \mathcal{C}_\Lambda = \mathcal{C}_{\Lambda^{*_t}})$,

$$(q_\infty^t)^z = ((q_\infty^{*_t})^t)^z$$

and, therefore, by (8.27), that, if Λ (respectively, Λ^{*_t}) is not minimal in S_{v_∞}, then

$$(q_\infty)^z = (q_\infty^{*_t})^z.$$

By Lemma 8.7, we conclude, ultimately, that, unless Λ (respectively, Λ^{*_t}) is minimal in S_{v_∞}, Λ and Λ^{*_t} share conformal curvature line coordinates (recall the close relationship between q_∞ (respectively, $q_\infty^{*_t}$) and the Hopf differential of Λ (respectively, Λ^{*_t}), cf. observed in Section 8.2.1).

8.2.5. Isothermic Darboux transformation vs. constrained Willmore Bäcklund transformation of CMC surfaces in 3-space. U. Hertrich-Jeromin and F. Pedit [**33**] proved that, for special choices of parameters, the isothermic Darboux transformation of CMC surfaces in Euclidean 3-space preserves the constancy of the mean curvature and, furthermore, the mean curvature. This is also the case for constrained Willmore Bäcklund transformation of these surfaces, cf. Section 8.2.4. Moreover, as proven by S. Kobayashi and J.-I. Inoguchi [**35**], isothermic Darboux transformation of CMC surfaces in $\mathbb{R}^3$ is equivalent to Bianchi-Bäcklund transformation, the latter described in [**48**] in terms of a dressing action, following the work of Terng and Uhlenbeck [**54**], just like the constrained Willmore Bäcklund transformation presented above. How are these transformations related? And what to say for general 3-space? This shall be the subject of further work. In [**12**], a description of Darboux transformation of constant mean curvature surfaces in Euclidean 3-space is presented in the quaternionic setting. It is based on the solution of a Riccati equation and it displays a striking similarity with the Darboux transformation of constrained Willmore surfaces in 4-space presented in Chapter 9 below. We prove that all non-trivial Darboux transforms of constrained Willmore surfaces can be obtained by constrained Wilmore Bäcklund transformation. We believe isothermic Darboux transformation of a CMC surface in Euclidean 3-space can be obtained as a particular case of constrained Willmore Bäcklund transformation.

CHAPTER 9

The special case of surfaces in 4-space

This chapter is dedicated to the special case of surfaces in 4-space. Our approach is quaternionic, based on the model of the conformal 4-sphere on the quaternionic projective space, and follows the work of F. Burstall et al. [**12**]. We identify $\mathbb{H}^2$ with $\mathbb{C}^4$ and provide $\wedge^2\mathbb{C}^4$ with the real structure $\wedge^2 j$ and with a certain metric inducing a metric with signature $(5,1)$ on the space of real vectors of $\wedge^2\mathbb{C}^4$. Via the Plücker embedding, we identify a j-stable 2-plane L in $\mathbb{C}^4$ with the real null line $\wedge^2 L$ in $(\mathrm{Fix}(\wedge^2 j))^{\mathbb{C}}$, presenting, in this way, the quaternionic projective space as a model for the conformal 4-sphere, $\mathbb{H}P^1 \cong S^4$. Surfaces in S^4 are described in this model as the immersed bundles of j-stable 2-planes in $\mathbb{C}^4$. We extend the Darboux transformation of Willmore surfaces in S^4 presented in [**12**], based on the solution of a Riccati equation, to a transformation of constrained Willmore surfaces in 4-space. We apply, yet again, the dressing action presented in Chapter 6 to define another transformation of constrained Willmore surfaces in 4-space, the *untwisted Bäcklund transformation*, referring then to the original one as the *twisted Bäcklund transformation*. We verify that, when both are defined, twisted and untwisted Bäcklund transformations coincide. We prove that constrained Willmore Darboux transformation of parameters ρ, T with $\rho > 1$ is equivalent to untwisted Bäcklund transformation of parameters α, L^α with α^2 real. Constrained Willmore Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial.

9.1. Surfaces in $S^4 \cong \mathbb{H}P^1$

Consider the natural identification of $\mathbb{H}$ with $\mathbb{R}^4$ and then the natural identification of $\mathbb{H}^2$ with $\langle 1, i\rangle^4 = \mathbb{C}^4$. Provide $\wedge^2\mathbb{C}^4$ with the real structure $\wedge^2 j$. Define a metric on $\wedge^2\mathbb{C}^4$ by $(v_1 \wedge v_2, v_3 \wedge v_4) := -\det(v_1, v_2, v_3, v_4)$, for $v_1, v_2, v_3, v_4 \in \mathbb{C}^4$, with $\det(v_1, v_2, v_3, v_4)$ denoting the determinant of the matrix whose columns are the components of v_1, v_2, v_3 and v_4, respectively, on the canonical basis of $\mathbb{C}^4$. This metric induces a metric with signature $(5,1)$ on the space of real vectors of $\wedge^2\mathbb{C}^4$, $\mathrm{Fix}(\wedge^2 j) = \mathbb{R}^{5,1}$. Via the Plücker embedding, we identify a j-stable 2-plane L in $\mathbb{C}^4$ with the real null line $\wedge^2 L$ in $(\mathrm{Fix}(\wedge^2 j))^{\mathbb{C}}$, presenting, in this way, the quaternionic projective space $\mathbb{H}P^1$ as a model for the conformal 4-sphere, and describing, in this model, surfaces in S^4 as the immersed bundles $L \cong \wedge^2 L : M \to S^4$ of j-stable 2-planes in $\mathbb{C}^4$.

9.1.1. Linear algebra. Consider the quaternions, the unitary $\mathbb{R}$-algebra $\mathbb{H}$ generated by i, j, k with the relations

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

We identify the real vector space $\mathbb{H}$ in the obvious way with $\mathbb{R}^4$: given $a, b, c, d \in \mathbb{R}$,

$$a + ib + jc + kd \cong (a, b, c, d);$$

and identify then $\mathbb{H}^2$ with $\mathbb{C}^4$ for $\mathbb{C} := \langle 1, i \rangle$, i.e., given $a, b, c, d, a', b', c', d' \in \mathbb{R}$,

$$((a, b, c, d), (a', b', c', d')) \cong (a + ia', b + ib', c + ic', d + id').$$

Under this identification, the left multiplication by the quaternionic unit j defines a complex anti-linear map $j : \mathbb{C}^4 \to \mathbb{C}^4$. The projective space $\mathbb{H}P^1$ of the quaternionic lines in $\mathbb{H}^2$ is, in this way, described as the set of the j-stable 2-planes in $\mathbb{C}^4$.

Remark 9.1. *A 2-plane U in $\mathbb{C}^4$ is either j-stable or complementary to jU: $U \cap jU$ is, obviously, j-stable, and, therefore,* $\mathrm{rank}_{\mathbb{C}}(U \cap jU) \in \{0, 2\}$.

Fix $\det \in \wedge^4(\mathbb{C}^4)^* \backslash \{0\}$ such that

$$\overline{\det \circ \wedge^4 j} = \det \tag{9.1}$$

and that

$$\det(v_1, v_2, jv_1, jv_2) > 0 \tag{9.2}$$

for $v_1, v_2, jv_1, jv_2 \in \mathbb{C}^4$ linearly independent. To see that such a det exists, first observe that condition (9.1) is equivalent to the reality of $\det(e_1, e_2, e_3, e_4)$, for $e_1, e_2, e_3 = je_1, e_4 = -je_2$ the canonical basis of $\mathbb{C}^4$. This makes clear the existence of det in $\wedge^4(\mathbb{C}^4)^* \backslash \{0\}$ satisfying condition (9.1), determined up to a (non-zero) real scale. On the other hand, for such a det, given $v_1, v_2, jv_1, jv_2 \in \mathbb{C}^4$ linearly independent, $\det(v_1, v_2, jv_1, jv_2)$ is a non-zero real number, whose sign does not, in fact, depend on the basis $\{v_1, v_2, jv_1, jv_2\}$ of $\mathbb{C}^4$. [1]

Consider $\wedge^2\mathbb{C}^4$ equipped with the real structure $\wedge^2 j$ and the metric defined by

$$(v_1 \wedge v_2, v_3 \wedge v_4) := \det(v_1, v_2, v_3, v_4),$$

for $v_1, v_2, v_3, v_4 \in \mathbb{C}^4$. Condition (9.1), equivalent to

$$\overline{(v_1 \wedge v_2, v_3 \wedge v_4)} = (\overline{v_1 \wedge v_2}, \overline{v_3 \wedge v_4}),$$

[1] The argument above shows that a possible choice of such a det is the one given by $\det(v_1 \wedge v_2, v_3 \wedge v_4) := - \mid v_1, v_2, v_3, v_4 \mid$, for $v_1, v_2, v_3, v_4 \in \mathbb{C}^4$, with $\mid v_1, v_2, v_3, v_4 \mid$ denoting the determinant of the matrix whose columns are the components of v_1, v_2, v_3 and v_4, respectively, on the canonical basis of $\mathbb{C}^4$.

for $v_1, v_2, v_3, v_4 \in \mathbb{C}^4$, ensures that $\wedge^2\underline{\mathbb{C}}^4$ induces a (real) metric in $\mathrm{Fix}\,(\wedge^2 j)$, which condition (9.2) ensures to have signature $(5,1)$, as we verify next.

Proposition 9.2. $\wedge^2\underline{\mathbb{C}}^4$ *induces in* $\mathrm{Fix}\,(\wedge^2 j)$ *a metric with signature* $(5,1)$.

We therefore write

$$\mathrm{Fix}\,(\wedge^2 j) = \mathbb{R}^{5,1}$$

and so

$$\wedge^2\mathbb{C}^4 = (\mathbb{R}^{5,1})^{\mathbb{C}}.$$

The proof of the proposition will follow a very important remark, presented next.

By definition of the metric on $\wedge^2\mathbb{C}^4$, the complementarity of 2-planes W and $\hat{W}$ in $\mathbb{C}^4$,

$$\mathbb{C}^4 = W \oplus \hat{W},$$

is equivalent to $(w_1 \wedge w_2, \hat{w}_1 \wedge \hat{w}_2) \neq 0$, for w_1, w_2 frame of W and $\hat{w}_1, \hat{w}_2$ frame of $\hat{W}$, or, equivalently,

$$\wedge^2 W \cap (\wedge^2\hat{W})^{\perp} = \{0\}. \tag{9.3}$$

Condition (9.3), in its turn, establishes the complementarity of $\wedge^2 W$ and $\wedge^2\hat{W}$ in $\wedge^2 W + \wedge^2\hat{W}$, together with the non-degeneracy of $\wedge^2 W \oplus \wedge^2\hat{W}$, ensuring, in particular, that

$$W \wedge \hat{W} = (\wedge^2 W \oplus \wedge^2\hat{W})^{\perp} \tag{9.4}$$

and establishing the decomposition

$$\wedge^2\,\mathbb{C}^4 = \wedge^2 W \oplus \wedge^2\hat{W} \oplus W \wedge \hat{W}. \tag{9.5}$$

Conversely, a decomposition

$$\wedge^2\,\mathbb{C}^4 = V \oplus V_+^{\perp} \oplus V_-^{\perp}, \tag{9.6}$$

with $V_+^{\perp}$ and $V_-^{\perp}$ null lines such that $V_+^{\perp} \cap (V_-^{\perp})^{\perp} = \{0\}$, and $V^{\perp} = V_+^{\perp} \oplus V_-^{\perp}$, determines complementary 2-planes W and $\hat{W}$ in $\mathbb{C}^4$ such that $\wedge^2 W = V_+^{\perp}$, $\wedge^2\hat{W} = V_-^{\perp}$ and $V = W \wedge \hat{W}$.

For the particular case of W a non-j-stable 2-plane and $\hat{W} = jW$, and in view of the reality of $W \wedge jW$ and $\wedge^2 W \oplus \wedge^2 jW$, (9.5) gives, in particular,

$$\mathrm{Fix}(\wedge^2 j) = (\wedge^2 W \oplus \wedge^2 jW) \cap \mathrm{Fix}(\wedge^2 j) \oplus (W \wedge jW) \cap \mathrm{Fix}(\wedge^2 j).$$

We now prove Proposition 9.2.

Proof. Let W be a non-j-stable 2-plane in $\mathbb{C}^4$. Given $w_i \in W$, for $i = 1,2,3,4$, the vector $w_1 \wedge w_2 + jw_3 \wedge jw_4$ is real if and only if $w_1 \wedge w_2 = w_3 \wedge w_4$. Hence, if $w_1 \wedge w_2 + jw_3 \wedge jw_4$ is non-zero, then $w_3 \wedge w_4 \neq 0$ and $w_i = \lambda_i^1 w_1 + \lambda_i^2 w_2$, for $i = 3,4$.

In that case,

$$(w_1 \wedge w_2 + jw_3 \wedge jw_4, w_1 \wedge w_2 + jw_3 \wedge jw_4) = 2\,\overline{\lambda_3^1\lambda_4^2 - \lambda_4^1\lambda_3^2}\det(w_1, w_2, jw_1, jw_2),$$

whilst

$$0 \neq jw_3 \wedge jw_4 = \overline{\lambda_3^1\lambda_4^2 - \lambda_4^1\lambda_3^2}\,jw_1 \wedge jw_2 = \overline{\lambda_3^1\lambda_4^2 - \lambda_4^1\lambda_3^2}\,jw_3 \wedge jw_4$$

and, therefore, $\overline{\lambda_3^1\lambda_4^2 - \lambda_4^1\lambda_3^2} = 1$. By equation (9.2), we conclude that, for non-zero real $w_1 \wedge w_2 + jw_3 \wedge jw_4 \in \wedge^2 W \oplus \wedge^2 jW$,

$$(w_1 \wedge w_2 + jw_3 \wedge jw_4, w_1 \wedge w_2 + jw_3 \wedge jw_4) = 2\det(w_1, w_2, jw_1, jw_2) > 0,$$

i.e., that the metric induced in $(\wedge^2 W \oplus \wedge^2 jW) \cap \mathrm{Fix}(\wedge^2 j)$ is positive definite. Now let w_1, w_2 be a basis of W, so that $w_1 \wedge jw_1, w_1 \wedge jw_2, w_2 \wedge jw_1$ and $w_2 \wedge jw_2$ form a basis of $W \wedge jW$. The vectors $w_1 \wedge jw_1$ and $w_2 \wedge jw_2$ are real, null and not orthogonal, spanning, therefore, a subspace of $(W \wedge jW) \cap \mathrm{Fix}\,(\wedge^2 j)$ with signature $(1,1)$. On the other hand, $\langle w_1 \wedge jw_2, w_2 \wedge jw_1\rangle \cap \mathrm{Fix}\,(\wedge^2 j) = \{(a+ib)w_1 \wedge jw_2 + (a-ib)w_2 \wedge jw_1 : a, b \in \mathbb{R}\}$, clearly orthogonal to $\langle w_1 \wedge jw_1, w_2 \wedge jw_2\rangle$ and with positive definite metric inherited from $\wedge^2\mathbb{C}^4$: given $w := (a+ib)w_1 \wedge jw_2 + (a-ib)w_2 \wedge jw_1$ with $a+ib \in \mathbb{C}\backslash\{0\}$,

$$(w, w) = 2\,(a^2 + b^2)\det(w_1, w_2, jw_1, jw_2) > 0.$$

By (9.4), we conclude the existence of an orthogonal basis of $\mathrm{Fix}\,(\wedge^2 j)$ composed by five space-like vectors and one time-like vector. $\square$

Remark 9.3. *In the proof of Proposition 9.2, we have observed, in particular, that, given W a non-j-stable 2-plane in $\mathbb{C}^4$, the metric induced in the space of the real vectors in $\wedge^2 W \oplus \wedge^2 jW$ is positive definite. Hence, given $v \in \wedge^2 W$,*

$$(v, \overline{v}\,) = \frac{1}{2}\,(v + \overline{v}, v + \overline{v}) \geq 0,$$

vanishing if and only if $v = 0$. It follows, in particular, that, given $v \in \wedge^2\mathbb{C}^4$ decomposable,

$$(v, \overline{v}\,) \geq 0. \tag{9.7}$$

We identify 2-planes in $\mathbb{C}^4$ with null lines in $\wedge^2\mathbb{C}^4$, spanned by a decomposable vector, via the famous Plücker embedding, i.e., via the correspondence

$$L = \langle u, v\rangle \leftrightarrow \langle u \wedge v\rangle = \wedge^2 L,$$

given complex linearly independent u and v in $\mathbb{C}^4$. In this way, we identify, in particular, j-stable 2-planes in $\mathbb{C}^4$ with real null lines in $\wedge^2\mathbb{C}^4$ and then, naturally, with null lines in $\mathrm{Fix}(\wedge^2 j)$. This presents the quaternionic projective space as a model for the conformal 4-sphere.

Proposition 9.4.

$$\mathbb{H}P^1 \cong S^4.$$

This chapter is dedicated to the special case of surfaces in S^4, described in this model as the immersed bundles

$$L \cong \wedge^2 L : M \to S^4$$

of j-stable 2-planes in $\mathbb{C}^4$.

We use $\mathrm{End}_j(\mathbb{C}^4)$ to denote the set of endomorphisms of $\mathbb{C}^4$ commuting with j, and $\mathrm{Gl}_j(\mathbb{C}^4)$ (respectively, $sl_j(\mathbb{C}^4)$) to denote the invertible (respectively, trace-free) j-commuting endomorphisms of $\mathbb{C}^4$.

Lemma 9.5. *Given $\xi \in \mathrm{End}_j(\mathbb{C}^4)$ with $\xi^2 = aI$, for some $a \in \mathbb{R}$, if $a \geq 0$, then*

$$\xi = \sqrt{a}\, I,$$

for one of the square roots of a; whereas, if $a < 0$, then, given a choice of $\sqrt{a}$,

$$\xi = I \begin{cases} \sqrt{a} & \text{on } W \\ \overline{\sqrt{a}} = -\sqrt{a} & \text{on } jW \end{cases},$$

for some non-j-stable 2-plane W in $\mathbb{C}^4$.

Proof. As a complex endomorphism, ξ admits at least one complex eigenvalue. Given $\lambda \in \mathbb{C}$ an eigenvalue of ξ and u an eigenvector of ξ associated to λ,

$$au = \xi^2 u = \xi(\lambda u) = \lambda \xi u = \lambda^2 u$$

and, therefore, $\lambda = \pm\sqrt{a}$. On the other hand,

$$\xi(ju) = j\xi u = j\lambda u = \overline{\lambda} ju,$$

showing that $\overline{\lambda}$ is an eigenvalue of ξ, as well, and that, for E_λ and $E_{\overline{\lambda}}$ the eigenspaces associated to λ and $\overline{\lambda}$, respectively, we have $jE_\lambda \subset E_{\overline{\lambda}}$, or, equivalently, in view of the symmetry of roles between λ and $\overline{\lambda}$, $jE_\lambda = E_{\overline{\lambda}}$. It follows, in particular, that, if $a < 0$, and, therefore, $\sqrt{a}$ is purely imaginary, $\overline{\sqrt{a}} = -\sqrt{a}$, the eigenvalues of ξ are exactly $\sqrt{a}$ and $-\sqrt{a}$ and, as ξ is diagonalizable, $\underline{\mathbb{C}}^4 = E_{\sqrt{a}} \oplus jE_{\sqrt{a}}$. If $a > 0$, and so $\sqrt{a}$ is real, $-\sqrt{a}$ is not an eigenvalue of ξ and, therefore, $E_{\sqrt{a}} = \underline{\mathbb{C}}^4$, completing the proof. $\square$

According to the previous lemma, given $S \in \mathrm{End}_j(\mathbb{C}^4)$ with $S^2 = -I$, we have a decomposition

$$\mathbb{C}^4 = S_+ \oplus S_-,$$

for S_+ and $S_- = jS_+$ the eigenspaces of S associated to i and $-i$, respectively, a notation that will be kept throughout the chapter.

Remark 9.6. *Given $S \in \mathrm{End}_j(\mathbb{C}^4)$, such that $S^2 = -I$, and $v_\pm \in \wedge^2 S_\pm$, we have $(v_\pm, \overline{v_\pm}) \geq 0$, vanishing if and only if $v_\pm = 0$, respectively.*

We define a 2-*sphere in* $\wedge^2\mathbb{C}^4$ to be the complexification of a 2-sphere in $\mathbb{R}^{5,1}$. Recall that the 2-spheres in $\mathbb{R}^{5,1}$ are described as the non-degenerate 4-planes in $\mathbb{R}^{5,1}$.

Proposition 9.7. *The* 2*-spheres in* $\wedge^2\mathbb{C}^4$ *are described as the (real non-degenerate)* 4*-planes* $S_+ \wedge S_-$ *for* $S \in \mathrm{End}_j(\mathbb{C}^4)$ *such that* $S^2 = -I$*, determined up to sign.*

Proof. The complementarity in $\mathbb{C}^4$ of the 2-planes S_+ and S_- ensures that $S_+ \wedge S_-$ is a 4-plane, as well as, by recalling (9.4) and (9.5), its non-degeneracy in $\wedge^2\mathbb{C}^4$. The reality of $S_+ \wedge S_-$ is immediate from the fact that S_+ and S_- are intertwined by j. Conversely, a real non-degenerate 4-plane V in $\wedge^2\mathbb{C}^4$ determines, up to sign, a j-commuting endomorphism S of $\mathbb{C}^4$, such that $S^2 = -I$, for which $V = S_+ \wedge S_-$. Indeed, for such a V, $V^\perp$ is a real non-degenerate complex 2-plane, admitting, therefore, a unique decomposition $V^\perp = V^\perp_+ \oplus V^\perp_-$. as the direct sum of two null complex lines, complex conjugate of each other. The corresponding decomposition (9.6) determines a 2-plane W in $\mathbb{C}^4$ for which $\wedge^2 W = V^\perp_+$ and $V = W \wedge jW$. By

$$S := I\begin{cases} i & \text{on } W \\ -i & \text{on } jW \end{cases},$$

we define a j-commuting endomorphism S of $\mathbb{C}^4$ such that $S^2 = -I$, determined by V up to sign, for which $V = S_+ \wedge S_-$. □

Throughout this chapter, we will use the standard identification

$$sl(\mathbb{C}^4) \cong o((\mathbb{R}^{5,1})^{\mathbb{C}}),$$

of the special linear algebra $sl(\mathbb{C}^4)$ with the orthogonal algebra $o(\wedge^2\mathbb{C}^4)$, given by $f \mapsto \hat{f}$ with

$$\hat{f}(x \wedge y) = (fx) \wedge y + x \wedge (fy), \tag{9.8}$$

for $f \in sl(\mathbb{C}^4)$, $\hat{f} \in o(\wedge^2\mathbb{C}^4)$ and $x, y \in \mathbb{C}^4$. Observe that, under this identification, the reality of an endomorphism of $(\mathbb{R}^{5,1})^{\mathbb{C}}$ corresponds to the j-commutativity of an endomorphism of $\mathbb{C}^4$.

We complete this section by presenting a last identification that shall be used throughout this chapter. Observe that by $g \mapsto \hat{g}$ with

$$\hat{g}(u \wedge v) = gu \wedge gv,$$

for $g \in \mathrm{Sl}(\mathbb{C}^4)$, $\hat{g} \in O((\mathbb{R}^{5,1})^{\mathbb{C}})$ and $u, v \in \mathbb{C}^4$, we define a $2:1$ mapping,

$$g, -g \mapsto \hat{g},$$

of the special linear group $\mathrm{Sl}(\mathbb{C}^4)$ onto the orthogonal group $O((\mathbb{R}^{5,1})^{\mathbb{C}})$. For an orthogonal transformation

$$\xi = I\begin{cases} a & \text{on } \wedge^2 W \\ 1 & \text{on } W\wedge\hat{W} \\ a^{-1} & \text{on } \wedge^2 \hat{W} \end{cases} \in O((\mathbb{R}^{5,1})^{\mathbb{C}}),$$

with W and $\hat{W}$ complementary 2-planes in $\mathbb{C}^4$ and $a \in \mathbb{C}\backslash\{0\}$, and fixing a choice of $\sqrt{a}$, we will still denote by ξ the special linear transformation

$$\xi = I\begin{cases} \sqrt{a} & \text{on } W \\ \sqrt{a}^{\,-1} & \text{on } \hat{W} \end{cases} \in \mathrm{Sl}(\mathbb{C}^4),$$

defined up to sign depending on the choice of $\sqrt{a}$. Obviously,

$$\xi(u\wedge v) = \xi u \wedge \xi v,$$

for $u, v \in \mathbb{C}^4$, independently of the choice of $\sqrt{a}$. Observe that, in the particular case W is a non-j-stable 2-plane and $\hat{W} = jW$, ξ and $\overline{\xi}$ are related via j by

$$\xi \circ j = j \circ \overline{\xi}. \tag{9.9}$$

9.1.2. The mean curvature sphere congruence. We present the mean curvature sphere congruence and the Hopf fields of a surface in S^4, as defined in [**12**].

Let L be an immersed bundle of j-stable 2-planes in $\mathbb{C}^4$. Consider M provided with the conformal structure induced by the surface $\wedge^2 L : M \to S^4$. Given $S \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ such that

$$SL = L, \tag{9.10}$$

we define a section of $\mathrm{End}(\underline{\mathbb{C}}^4/L)$, which we still denote by S, by $S(x+L) := Sx + L$, for all $x \in \Gamma(\underline{\mathbb{C}}^4)$. Consider the derivative

$$\delta := \pi_{\underline{\mathbb{C}}^4/L} \circ d_{|L},$$

for $\pi_{\underline{\mathbb{C}}^4/L} : \underline{\mathbb{C}}^4 \to \underline{\mathbb{C}}^4/L$ the canonical projection.

Remark 9.8. *Consider the projection $\pi : \mathcal{L} \to \mathbb{P}(\mathcal{L})$ for the light-cone $\mathcal{L}$ in $\mathbb{R}^{5,1}$. Given l a never-zero section of $L \simeq \wedge^2 L : M \to \mathbb{P}(\mathcal{L})$, $d\pi_l$ gives an isomorphism $L^\perp/L \to T_L\mathbb{P}(\mathcal{L})$ under which the derivative of L, $dL = d\pi_l \circ dl = d\pi_l \circ \delta l$, is given by δl,*

$$dL = \delta l.$$

Therefore L immerses if and only if $\operatorname{rank} \delta L(TM) = 2$.

As presented in [**12**],

Definition 9.9. *The* mean curvature sphere (congruence) *of L is defined to be the unique j-commuting complex structure S on $\underline{\mathbb{C}}^4$ such that*

$$SL = L, \quad (dS)L \subset L, \quad *\,\delta = S \circ \delta = \delta \circ S_{|L}$$

and

$$(SdS - *dS)L = 0. \tag{9.11}$$

Once and for all, fix S as the mean curvature sphere of L. Decompose the trivial flat connection on $\underline{\mathbb{C}}^4$ as $d = \mathcal{D}_S + \mathcal{N}_S$ by the conditions

$$\mathcal{D}_S S = 0, \;\; \mathcal{N}_S S = -S\mathcal{N}_S. \tag{9.12}$$

Equivalently, set $\mathcal{N}_S := \frac{1}{2}SdS$ and $\mathcal{D}_S := d - \mathcal{N}_S$. Note that conditions (9.12) are described, equivalently, by

$$\mathcal{D}_S\Gamma(S_\pm) \subset \Omega^1(S_\pm), \tag{9.13}$$

together with

$$\mathcal{N}_S S_\pm \subset S_\mp \tag{9.14}$$

respectively. Consider the congruence $V = S_+ \wedge S_-$ of 2-spheres in $\wedge^2\underline{\mathbb{C}}^4$. Let $\pi_{\wedge^2 S_+ \oplus \wedge^2 S_-}$ denote the orthogonal projection of

$$\wedge^2\, \underline{\mathbb{C}}^4 = \wedge^2 S_+ \oplus S_+ \wedge S_- \oplus \wedge^2 S_- \tag{9.15}$$

onto $\wedge^2 S_+ \oplus \wedge^2 S_-$. The trivial flat connection on $\wedge^2\underline{\mathbb{C}}^4$ relates to the trivial flat connection on $\underline{\mathbb{C}}^4$ by

$$d(u \wedge v) = (du) \wedge v + u \wedge (dv),$$

for $u, v \in \Gamma(\underline{\mathbb{C}}^4)$. In particular, given $u \in \Gamma(S_+), v \in \Gamma(S_-)$,

$$\begin{aligned} \mathcal{N}_V(u \wedge v) &= (\pi_V \circ d \circ \pi_{V^\perp} + \pi_{V^\perp} \circ d \circ \pi_V)(u \wedge v) \\ &= \pi_{\wedge^2 S_+ \oplus \wedge^2 S_-}((du) \wedge v + u \wedge (dv)) \\ &= \pi_{\wedge^2 S_+ \oplus \wedge^2 S_-}((\mathcal{D}_S u) \wedge v + (\mathcal{N}_S u) \wedge v + u \wedge (\mathcal{D}_S v) + u \wedge (\mathcal{N}_S v)) \end{aligned}$$

and, therefore, according to (9.13) and (9.14),

$$\mathcal{N}_V(u \wedge v) = (\mathcal{N}_S u) \wedge v + u \wedge (\mathcal{N}_S v).$$

Similarly, we get to the same conclusion for $u \wedge v \in \Gamma(\wedge^2 S_+)$ and $u \wedge v \in \Gamma(\wedge^2 S_-)$. We conclude that $\mathcal{N}_V \cong \mathcal{N}_S$, under the identification defined by (9.8). In particular,

$$\mathcal{N}_S j = j\mathcal{N}_S.$$

We define a decomposition

$$\mathcal{N}_S = A + Q$$

with $A, Q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ given by

$$*A = SA \ , \ *Q = -SQ,$$

said to be the *Hopf fields* of L, following the terminology of [**12**], as we verify later in this section. Clearly,

$$A = \frac{1}{2}(\mathcal{N}_S - S * \mathcal{N}_S), \quad Q = \frac{1}{2}(\mathcal{N}_S + S * \mathcal{N}_S),$$

and, therefore, having in consideration (9.12), both A and Q anti-commute with S,

$$AS = -SA \ , \ QS = -SQ.$$

In particular, this makes clear that

$$AS_\pm \subset S_\mp, \quad QS_\pm \subset S_\mp, \tag{9.16}$$

respectively. On the other hand, $iA^{1,0} = -*A^{1,0} = -SA^{1,0}$, showing that $\mathrm{Im}\, A^{1,0} \subset S_-$ and, therefore, that

$$A^{1,0}S_- = 0. \tag{9.17}$$

Analogously, we verify that

$$A^{0,1}S_+ = Q^{1,0}S_+ = Q^{0,1}S_- = 0. \tag{9.18}$$

Note that, as

$$dS = 2\mathcal{N}_S S, \tag{9.19}$$

we have $Q = \frac{1}{4}\,(SdS - *dS)$. Thus

$$QL = 0.$$

Lastly, observe that

$$A^{1,0}s_+ = \frac{1}{2}\,(\mathcal{N}_S^{1,0}s_+ - iS\mathcal{N}_S^{1,0}s_+) = \mathcal{N}_S^{1,0}s_+ \in \Omega^{1,0}(L_-),$$

for $s_+ \in \Gamma(S_+)$,

$$A^{1,0}S_+ \subset L_-; \tag{9.20}$$

and, similarly,

$$A^{0,1}S_- \subset L_+$$

and

$$Q^{1,0}S_- \subset S_+ \ , \ Q^{0,1}S_+ \subset S_-.$$

We complete this section by verifying that our description follows the description presented in [**12**]. Given $\psi \in \Gamma(\underline{\mathbb{C}}^4)$, $d(S\psi) = -2S\mathcal{N}_S\psi + S\mathcal{D}_S\psi + S\mathcal{N}_S\psi = -S\mathcal{N}_S\psi + S\mathcal{D}_S\psi$ and, therefore, $S(d(S\psi)) = \mathcal{N}_S\psi - \mathcal{D}_S\psi$, as well as $*\,d(S\psi) = -S * \mathcal{N}_S\psi + S * \mathcal{D}_S\psi$. Thus $Q\psi = \frac{1}{4}(d\psi + S * d\psi + S(d(S\psi)) - *\,d(S\psi))$, showing that Q coincides with the

one defined in [**12**]. The similar conclusion with respect to A follows then from the fact that

$$dS = 2(*Q - *A), \tag{9.21}$$

made clear by (9.19).

9.1.3. Mean curvature sphere congruence and central sphere congruence. In codimension 2, the complexification of the normal $S^{\perp}$ to the central sphere congruence of a surface Λ admits a unique decomposition $S^{\perp} = S^{\perp}_{+} \oplus S^{\perp}_{-}$ into the direct sum of two null complex lines, complex conjugate of each other. Via the Plücker embedding, we identify $S^{\perp}_{+}$ with some bundle S_{+} of 2-planes in $\mathbb{C}^4$, and write then $S = S_{+} \wedge jS_{+}$. The mean curvature sphere of $L \cong \wedge^2 L = \Lambda$ is defined, up to sign, as the j-commuting complex structure on $\underline{\mathbb{C}}^4$ admitting S_{+} as the eigenspace associated to the eigenvalue i (and, therefore, jS_{+} as the eigenspace associated to $-i$). This establishes the close relationship between mean curvature sphere congruence and the central sphere congruence.

Let L be an immersed bundle of j-stable 2-planes in $\mathbb{C}^4$ and consider the surface

$$\Lambda := \wedge^2 L : M \to S^4.$$

Let S be the mean curvature sphere of L.

Proposition 9.10. *The congruence $V := S_{+} \wedge S_{-}$, of 2-spheres in $\wedge^2\mathbb{C}^4$, is the complexification of the central sphere congruence of Λ.*

Before proceeding to the proof of the proposition, time for a few considerations.

Equation (9.10) ensures that, given $l_i = l_i^{+} + l_i^{-} \in \Gamma(L)$ with $l_i^{\pm} \in \Gamma(S_{\pm})$, respectively, for $i = 1, 2$, l_1^{+} and l_2^{+} are linearly dependent, and so are l_1^{-} and l_2^{-}. We conclude the existence of a frame of L composed by a section of S_{+} and a section of S_{-}, which provides a decomposition

$$L = L_{+} \oplus L_{-}$$

of L into a sum of complex line bundles L_{+} and L_{-} with

$$L_{-} = jL_{+},$$

namely,

$$L_{+} := L \cap S_{+} \ , \ L_{-} := L \cap S_{-}.$$

Obviously,

$$\wedge^2 L = L_{+} \wedge L_{-}. \tag{9.22}$$

Now we proceed to the proof of Proposition 9.10.

Proof. By (9.22), $\wedge^2 L \subset S_+ \wedge S_-$. Equations (9.10) and (9.19), together with the fact that $(dS)L \subset L$, show that $\mathcal{N}_S L \subset L$ and, therefore, by (9.14), that

$$\mathcal{N}_S L_\pm \subset L_\mp, \tag{9.23}$$

respectively. Provide M with a conformal structure $\mathcal{C}$. In view of (9.23) and (9.10), given $l = l_+ + l_-$ in $\Gamma(L)$, with $l_+ \in \Gamma(L_+)$ and $l_- \in \Gamma(L_-)$, $S \circ \delta l = S(dl) + L = S(\mathcal{D}_S l_+ + \mathcal{D}_S l_-) + L$. Therefore, by (9.13), $S \circ \delta l = i(\mathcal{D}_S l_+ - \mathcal{D}_S l_-) + L$, whilst

$$*\delta l = -i(d^{1,0} l - d^{0,1} l) + L = -i(\mathcal{D}_S^{1,0} l - \mathcal{D}_S^{0,1} l) + L.$$

Thus $*\delta l = S \circ \delta l$ if and only if $\mathcal{D}_S^{1,0} l_+ - \mathcal{D}_S^{0,1} l_- \in \Gamma(L)$. By equation $*\,\delta = S \circ \delta$, we conclude that

$$\mathcal{D}_S^{1,0}\,\Gamma(L_+) \subset \Omega^{1,0}(L_+),$$

or, equivalently,

$$\mathcal{D}_S^{0,1}\,\Gamma(L_-) \subset \Omega^{0,1}(L_-).$$

It follows that, given z holomorphic chart of $(M, \mathcal{C}_\Lambda)$ and never-zero $l_+ \in \Gamma(L_+)$ and $l_- \in \Gamma(L_-)$,

$$\Lambda^{1,0} = \langle l_+ \wedge l_-, d_{\delta_z}(l_+ \wedge l_-)\rangle = \langle l_+ \wedge l_-, l_+ \wedge (\mathcal{D}_S)_{\delta_z} l_-\rangle,$$

showing, in particular, the linear independency of l_- and $(\mathcal{D}_S)_{\delta_z} l_-$, and, consequently, that

$$\Lambda^{1,0} = L_+ \wedge S_-$$

and, by complex conjugation,

$$\Lambda^{0,1} = L_- \wedge S_+.$$

In particular, it is clear that V envelops the surface Λ. Lastly, according to equation (9.11), we have, in particular, that, given $l_- \in \Gamma(L_-)$, $(Sd^{1,0}S + id^{1,0}S)l_- = 0$, or, equivalently, $(d^{1,0}S)l_- \in \Omega^{1,0}(S_-)$, which, in its turn, according to (9.14), is equivalent to $\mathcal{N}_S^{1,0} l_- = 0$. Hence

$$\mathcal{N}_S^{1,0} L_- = 0.$$

Similarly, we verify that

$$\mathcal{N}_S^{0,1} L_+ = 0.$$

On the other hand, according to (9.18) and (9.20), we have

$$\mathcal{N}_S^{1,0} S_+ \subset L_-$$

and, similarly,

$$\mathcal{N}_S^{0,1} S_- \subset L_+.$$

It follows that $\mathcal{N}_S^{1,0}(L_- \wedge S_+) = 0 = \mathcal{N}_S^{0,1}(L_+ \wedge S_-)$, i.e., $\mathcal{N}_V^{1,0}\Lambda^{0,1} = 0 = \mathcal{N}_V^{0,1}\Lambda^{1,0}$, V is central with respect to Λ, completing the proof. $\square$

Remark 9.11. *The mean curvature sphere of L is, conversely, determined by the complexification of the central sphere congruence of Λ, as follows. Since the complexification V of the central sphere congruence of Λ is a 2-sphere, $V = S_+ \wedge S_-$, for some $S \in \mathrm{End}_j(\mathbb{C}^4)$ such that $S^2 = -I$, determined up to sign. One of them is the mean curvature sphere of L, cf. Proposition 9.10. For the other one, define $\mathcal{D}_S$ analogously to the case of the mean curvature sphere. Of course,*

$$\mathcal{D}_{-S} = \mathcal{D}_S.$$

The conformality of sections of Λ, characterized by the isotropy of

$$d^{1,0}(l_+ \wedge l_-) = (\mathcal{D}_S^{1,0} l_+) \wedge l_- + l_+ \wedge (\mathcal{D}_S^{1,0} l_-),$$

or, equivalently,

$$(\mathcal{D}_S^{1,0} l_+) \wedge l_- \wedge l_+ \wedge (\mathcal{D}_S^{1,0} l_-) = 0,$$

for all $l_+ \in \Gamma(L_+)$ and $l_- \in \Gamma(L_-)$, amounts, according to (9.13), *to either*

$$\mathcal{D}_S^{1,0}\,\Gamma(L_+) \subset \Omega^{1,0}(L_+) \tag{9.24}$$

or

$$\mathcal{D}_S^{1,0}\,\Gamma(L_-) \subset \Omega^{1,0}(L_-),$$

depending on the sign of S. Equation (9.24) *determines S for which, in particular, given $l = l_+ + l_- \in \Gamma(L)$ with $l_\pm \in \Gamma(L_\pm)$ respectively, $\mathcal{D}_S^{1,0} l_+ - \mathcal{D}_S^{0,1} l_- \in \Gamma(L)$, or, equivalently, $*\,\delta = S \circ \delta$. S determined in this way is the mean curvature sphere of L.*

9.2. Constrained Willmore surfaces in 4-space

In this section, we characterize constrained Willmore surfaces in 4-space in the quaternionic setting. We present, in particular, a characterization of this class of surfaces in terms of the closeness of a certain form, following the characterization of the harmonicity of the mean curvature sphere presented in [**12**].

Let $L \cong \Lambda \subset \underline{\mathbb{R}}^{5,1}$ be a surface in S^4. Let S be the mean curvature sphere of L and V be the complexification of the central sphere congruence of Λ,

$$V = S_+ \wedge S_-,$$

with orthogonal complement

$$V^\perp = \wedge^2 S_+ \oplus \wedge^2 S_-.$$

Provide M with the conformal structure induced by Λ. Consider the map $j : \mathbb{C}^4/L \to \mathbb{C}^4/L$ induced naturally by the quaternionic structure j in $\mathbb{C}^4$, having in consideration that L is j-stable. Constrained Willmore surfaces in S^4 are, alternatively, characterized as follows:

Theorem 9.12. *An immersed bundle L of j-stable 2-planes in $\mathbb{C}^4$ is a constrained Willmore surface if and only if there exists $q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L))$ such that*

$$Sq = *\,q = qS \tag{9.25}$$

and

$$d^{\mathcal{D}_S} q = 0, \quad d^{\mathcal{D}_S} * \mathcal{N}_S = 2[q \wedge *\mathcal{N}_S].$$

In the proof of the theorem, we establish a correspondence between 1-forms with values in $\Lambda \wedge \Lambda^{(1)}$ and S-commuting 1-forms with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$, under the the identification $sl(\mathbb{C}^4) \cong o((\mathbb{R}^{5,1})^{\mathbb{C}})$ presented in Section 9.1.1. Under this correspondence, the conditions on q in Theorem 9.12 characterize q as a multiplier to Λ.

Next we prove Theorem 9.12.

Proof. First note that

$$\Lambda^{\perp} = \underline{\mathbb{C}}^4 \wedge L.$$

Let q be a 1-form with values in $o((\underline{\mathbb{R}}^{5,1})^{\mathbb{C}}) = sl(\underline{\mathbb{C}}^4)$. Suppose that $q\Lambda = 0$ and $q\Lambda^{\perp} \subset \Lambda$. Let l, l', u, v be a frame of $\underline{\mathbb{C}}^4$ with $l, l' \in \Gamma(L)$. The fact that q vanishes in Λ, equivalent to

$$(ql) \wedge l' + l \wedge (ql') = 0, \tag{9.26}$$

shows that ql has no component in $\langle u \rangle$ nor in $\langle v \rangle$. On the other hand, since $q\Lambda^{\perp} \subset \Lambda$, we have, in particular,

$$(qu) \wedge l + u \wedge (ql) \in \Gamma(\langle l \wedge l' \rangle), \tag{9.27}$$

forcing the component of ql in $\langle l' \rangle$ to vanish. Hence $ql \in \Gamma(\langle l \rangle)$, and, by symmetry of roles, $ql' \in \Gamma(\langle l' \rangle)$. Equation (9.26) shows now that, if $ql = al$, with $a \in \Gamma(\underline{\mathbb{C}})$, then $ql' = -al'$. By the skew-symmetry of q, it follows that

$$-a^2(l \wedge l', u \wedge v) = -(l \wedge l', qu \wedge qv).$$

Equation (9.27) forces, on the other hand, the component of qu in $\langle v \rangle$ to vanish. Similarly, the fact that

$$(qv) \wedge l + v \wedge (ql) \in \Gamma(\langle l \wedge l' \rangle)$$

shows, in particular, that the component of qv in $\langle u \rangle$ vanishes. Thus $a = 0$. We conclude that $qL = 0$. Yet again by $q\Lambda^{\perp} \subset \Lambda$, it follows that $\mathrm{Im}\, q \subset L$. It is clear that, conversely, if $qL = 0$ and $\mathrm{Im}\, q \subset L$, then $q\Lambda = 0$ and $q\Lambda^{\perp} \subset \Lambda$. We conclude that q takes values in $\Lambda \wedge \Lambda^{\perp}$, or, equivalently, $q\Lambda = 0$ and $q\Lambda^{\perp} \subset \Lambda$, if and only if

$$qL = 0, \ \ \mathrm{Im}\, q \subset L,$$

i.e., q defines a 1-form with values in $\mathrm{End}(\underline{\mathbb{C}}^4/L, L)$. Note that

$$\Lambda \wedge \Lambda^{(1)} = \Lambda \wedge \Lambda^{\perp} \cap (\wedge^2 V \oplus \wedge^2 V^{\perp}).$$

Now observe that if q preserves $S_+ \wedge S_-$ then, given $s_+ \in \Gamma(S_+)$ and $s_- \in \Gamma(S_+)$, the component of qs_+ in $\langle s_- \rangle$ vanishes, as well as the component of qs_- in $\langle s_+ \rangle$ does, which amounts to q preserving both S_+ and S_-. It is clear that, conversely, if q preserves both S_+ and S_-, then q preserves $S_+ \wedge S_-$, $\wedge^2 S_+$ and $\wedge^2 S_-$. Hence q takes values in $(\wedge^2 V \oplus \wedge^2 V^\perp)$, or, equivalently, q preserves both V and $V^\perp$, if and only if q preserves the eigenspaces of S. In its turn, q preserves $S_\pm$ if and only if q commutes with S in $S_\pm$, respectively.

Now recall Lemma 5.10, establishing that, if $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ is real and $d^{\mathcal{D}_V} q$ vanishes, then $q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1})$, or, equivalently, $q^{0,1} \in \Omega^{0,1}(\Lambda \wedge \Lambda^{1,0})$. In that case, $q^{1,0}(L_- \wedge S_+) = 0 = q^{0,1}(L_+ \wedge S_-)$, or, equivalently, $L_- \wedge q^{1,0} S_+ = 0 = L_+ \wedge q^{0,1} S_-$, and, therefore,

$$q^{1,0} S_+ = 0 = q^{0,1} S_-,$$

as $q^{1,0} S_+ \subset L_+$ and $q^{0,1} S_- \subset L_-$. Hence

$$\operatorname{Im} q^{1,0} \subset S_-, \quad \operatorname{Im} q^{0,1} \subset S_+,$$

showing that

$$Sq^{1,0} = -iq^{1,0} = *\, q^{1,0}, \quad Sq^{0,1} = iq^{0,1} = *\, q^{0,1}$$

and, ultimately, that $Sq = *\, q$, completing the proof. □

Following the characterization of the harmonicity of S presented in [**12**], we have, more generally:

Theorem 9.13. *Suppose $q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L))$ satisfies condition* (9.25)*. An immersed bundle L of j-stable 2-planes in $\mathbb{C}^4$ is a q-constrained Willmore surface if and only if either of the following equations is verified:*

i) $d * \mathcal{N}_S + 2\, d *\, q = 0$;

ii) $d * (A + q) = 0$;

iii) $d * (Q + q) = 0$.

Before proceeding to the proof of the theorem, observe that equation (9.21) establishes, in particular,

$$d * A = d *\, Q. \tag{9.28}$$

Now we prove Theorem 9.13.

Proof. In view of (9.25),

$$d^{\mathcal{D}_S} *\, q = d^{\mathcal{D}_S} Sq = (\mathcal{D}_S S) \wedge q + S d^{\mathcal{D}_S} q = S d^{\mathcal{D}_S} q, \tag{9.29}$$

and, therefore, we have $d *\, q = [\mathcal{N}_S \wedge *\, q] = -[q \wedge *\mathcal{N}_S]$ if and only if $d^{\mathcal{D}_S} q = 0$. In particular, if L is a constrained Willmore surface, then

$$d * \mathcal{N}_S + 2\, d *\, q = d^{\mathcal{D}_S} * \mathcal{N}_S + [\mathcal{N}_S \wedge *\mathcal{N}_S] - 2[q \wedge *\mathcal{N}_S] = 0.$$

Yet again in view of (9.25), $(d^{\mathcal{D}_S} * q)S = d^{\mathcal{D}_S}(*qS) + *q \wedge (\mathcal{D}_S S) = -d^{\mathcal{D}_S}q$, whereas, following (9.29), $S\, d^{\mathcal{D}_S} * q = -d^{\mathcal{D}_S}q$. Thus

$$S\, d^{\mathcal{D}_S} * q = (d^{\mathcal{D}_S} * q)S.$$

Similarly, as

$$*\mathcal{N}_S S = -Q + A = -S * \mathcal{N}_S, \tag{9.30}$$

we have

$$(d^{\mathcal{D}_S} * \mathcal{N}_S)S = d^{\mathcal{D}_S}(*\mathcal{N}_S S) + *\mathcal{N}_S \wedge (\mathcal{D}_S S) = d^{\mathcal{D}_S}(-Q + A),$$

whereas

$$d^{\mathcal{D}_S} *\mathcal{N}_S = d^{\mathcal{D}_S}(S(-Q+A)) = (\mathcal{D}_S S) \wedge (-Q+A) + S\, d^{\mathcal{D}_S}(-Q+A) = S\, d^{\mathcal{D}_S}(-Q+A),$$

and, therefore,

$$S\, d^{\mathcal{D}_S} * \mathcal{N}_S = -(d^{\mathcal{D}_S} * \mathcal{N}_S)S.$$

On the other hand, together, (9.25) and (9.30) establish

$$[\mathcal{N}_S \wedge *q]S = -S[\mathcal{N}_S \wedge *q].$$

It follows that, if equation $i)$ holds, $0 = d^{\mathcal{D}_S} * \mathcal{N}_S + 2d^{\mathcal{D}_S} * q + 2[\mathcal{N}_S \wedge *q]$, then, equivalently, $d^{\mathcal{D}_S} * \mathcal{N}_S + 2[\mathcal{N}_S \wedge *q] = 0 = 2d^{\mathcal{D}_S} * q$, by separating S-commuting and S-anti-commuting parts.

Equation (9.28) establishes the equivalence of $i)$ and either equations $ii)$ or $iii)$, completing the proof. $\square$

We complete this section with an approach to constrained Willmore surfaces in 4-space in terms of flatness of connections. Let $q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L))$. For each $\lambda \in \mathbb{C}\backslash\{0\}$, define a connection on $\underline{\mathbb{C}}^4$ by

$$d_S^{\lambda,q} := \mathcal{D}_S + \lambda^{-1}\mathcal{N}_S^{1,0} + \lambda\mathcal{N}_S^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1}.$$

For each λ, the connection $d_V^{\lambda,q}$, on $\wedge^2\underline{\mathbb{C}}^4$, relates to $d_S^{\lambda,q}$ via equation (9.8), and so do, therefore, the respective curvature tensors,

$$R^{d_V^{\lambda,q}}(\sigma_1 \wedge \sigma_2) = (R^{d_S^{\lambda,q}}\sigma_1) \wedge \sigma_2 + \sigma_1 \wedge (R^{d_S^{\lambda,q}}\sigma_2),$$

for $\sigma_1, \sigma_2 \in \Gamma(\underline{\mathbb{C}}^4)$. Hence L is q-constrained Willmore if and only if $d_S^{\lambda,q}$ is flat, for all $\lambda \in \mathbb{C}\backslash\{0\}$. For $\lambda \in \mathbb{C}\backslash\{0\}$, define an orthogonal transformation of $\wedge^2\underline{\mathbb{C}}^4$ by

$$\tau(\lambda) := I \begin{cases} \lambda^{-1} & \text{on } \wedge^2 S_+ \\ 1 & \text{on } S_+ \wedge S_- \\ \lambda & \text{on } \wedge^2 S_- \end{cases}.$$

Given $\lambda \in \mathbb{C}\backslash\{0\}$, we define a new connection on $\wedge^2\underline{\mathbb{C}}^4$ by

$$d^V_{\lambda^2,q} := \tau(\lambda) \circ d_V^{\lambda,q} \circ \tau(\lambda)^{-1};$$

as well as, fixing a choice of $\sqrt{\lambda}$, and, with respect to (and independently of) this choice, a new connection on $\underline{\mathbb{C}}^4$ by

$$d^S_{\lambda^2,q} := \tau(\lambda) \circ d_S^{\lambda,q} \circ \tau(\lambda)^{-1},$$

for $\tau \in \Gamma(Sl(\underline{\mathbb{C}}^4))$ as defined in Section 9.1.1. These definitions only depend, indeed, on λ^2, as, in particular, the next lemma makes clear (note that $d^V_{\lambda^2,q}$ is related to $d^S_{\lambda^2,q}$ by equation (9.8)). The curvature tensors of $d^V_{\lambda^2,q}$ and $d_V^{\lambda,q}$ are related by

$$R^{d^V_{\lambda^2,q}} = \tau(\lambda)\, R^{d_V^{\lambda,q}}\, \tau(\lambda)^{-1},$$

showing that the flatness of $d^V_{\lambda^2,q}$, or, equivalently, that of $d^S_{\lambda^2,q}$, for all $\lambda \in \mathbb{C}\backslash\{0\}$, provides another characterization of the q-constrained Willmore condition of L. It is obvious but it will be useful to remark that, for each λ, the $d^S_{\lambda^2,q}$-parallelness of a bundle $W \subset \underline{\mathbb{C}}^4$ is equivalent to the $d^V_{\lambda^2,q}$-parallelness of $\wedge^2 W \subset \wedge^2\underline{\mathbb{C}}^4$.

Lemma 9.14. *For each* $\lambda \in \mathbb{C}\backslash\{0\}$,

$$d^S_{\lambda^2,q} = d + (\lambda^{-2} - 1)(Q^{1,0} + q^{1,0}) + (\lambda^2 - 1)(Q^{0,1} + q^{0,1}).$$

Proof. Fix $\lambda \in \mathbb{C}\backslash\{0\}$. The fact that both $q^{1,0}$ and $q^{0,1}$ preserve S_+ and S_- establishes, in particular,

$$\tau(\lambda) \circ q \circ \tau(\lambda)^{-1} = q.$$

Similarly, in view of the $\mathcal{D}_S$-parallelness of S_+ and S_- (cf. (9.13)),

$$\tau(\lambda) \circ \mathcal{D}_S \circ \tau(\lambda)^{-1} = \mathcal{D}_S.$$

On the other hand, according to (9.16), (9.17) and (9.18),

$$\tau(\lambda) \circ A^{1,0} \circ \tau(\lambda)^{-1} = \lambda A^{1,0}, \quad \tau(\lambda) \circ A^{0,1} \circ \tau(\lambda)^{-1} = \lambda^{-1} A^{0,1}$$

and

$$\tau(\lambda) \circ Q^{1,0} \circ \tau(\lambda)^{-1} = \lambda^{-1} Q^{1,0}, \quad \tau(\lambda) \circ Q^{0,1} \circ \tau(\lambda)^{-1} = \lambda Q^{0,1}.$$

Hence

$$d^S_{\lambda^2,q} = \mathcal{D}_S + (\lambda^{-2} - 1)(Q^{1,0} + q^{1,0}) + (\lambda^2 - 1)(Q^{0,1} + q^{0,1}) + A + Q,$$

completing the proof. □

9.3. Transformations of constrained Willmore surfaces in 4-space

Under the standard identification $sl(\mathbb{C}^4) \cong o(\wedge^2\mathbb{C}^4)$, 1-forms with values in $\Lambda \wedge \Lambda^{(1)}$ correspond to S-commuting 1-forms with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$, for S the mean curvature sphere congruence of $L \cong \Lambda$. For such a form q, condition $d^{\mathcal{D}} q = 0$ establishes $Sq = *q$. A surface L in S^4 is a q-constrained Willmore surface, for some 1-form q with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$ such that $Sq = *\, q = qS$, if and only if $d * (Q + q) = 0$, for the Hopf field $Q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4))$. The closeness of the 1-form $*(Q + q)$ ensures the

existence of $G \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ with $dG = 2*(Q+q)$, as well as the integrability of the Riccati equation $dT = \rho T(dG)T - dF + 4\rho qT$, for each $\rho \in \mathbb{R}\backslash\{0\}$, fixing such a G and setting $F := G - S$. For a local solution $T \in \Gamma(Gl_j(\underline{\mathbb{C}}^4))$ of the ρ-Ricatti equation, we define the *constrained Willmore Darboux transform* of L of parameters ρ, T by setting $\hat{L} := T^{-1}L$, and extend, in this way, the Darboux transformation of Willmore surfaces in S^4 presented in [**12**] to a transformation of constrained Willmore surfaces in 4-space. We apply, yet again, the dressing action presented in Chapter 6 to define another transformation of constrained Willmore surfaces in 4-space, the *untwisted Bäcklund transformation*, referring then to the original one as the *twisted Bäcklund transformation*. We verify that, when both are defined, twisted and untwisted Bäcklund transformations coincide. We prove that constrained Willmore Darboux transformation of parameters ρ, T with $\rho > 1$ is equivalent to untwisted Bäcklund transformation of parameters α, L^α with α^2 real. Constrained Willmore Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial.

Let $L \cong \Lambda \subset \underline{\mathbb{R}}^{5,1}$ be a q-constrained Willmore surface in S^4, for some 1-form q with values in $\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L)$. Let S be the mean curvature sphere of L and V be the complexification of the central sphere congruence of Λ. Let ρ_V be reflection across V. Provide M with the conformal structure induced by Λ. For simplicity, in what follows, given $\alpha \in \mathbb{C}\backslash S^1$ non-zero and L^α a $d_V^{\alpha,q}$-parallel null line subbundle of $\wedge^2\underline{\mathbb{C}}^4$ such that, locally,

$$\rho_V L^\alpha \cap (L^\alpha)^\perp = \{0\} = \rho_V \tilde{L}^\alpha \cap (\tilde{L}^\alpha)^\perp, \tag{9.31}$$

we consider the alternative notation

$$p := p_{\alpha,\tilde{L}^\alpha}, \;\; q := q_{\overline{\alpha}^{-1},\overline{L^\alpha}},$$

with no risk of ambiguity with the multiplier to Λ. Recall that, for such α and L^α, we define another constrained Willmore surface in S^4, the Bäcklund transform Λ^* of Λ of parameters α, L^α, or, equally, $-\alpha, \rho_V L^\alpha$, by setting

$$(\Lambda^*)^{0,1} := (pq)^{-1}(1)(pq)(0)\Lambda^{0,1}, \tag{9.32}$$

provided that Λ^* immerses.

9.3.1. Untwisted Bäcklund transformation of constrained Willmore surfaces in 4-space. In this section, we apply, yet again, the dressing action presented in Section 6.5 to define another transformation of constrained Willmore surfaces in 4-space, the *untwisted Bäcklund transformation*, referring then to the Bäcklund transformation defined by (9.32) as the *twisted Bäcklund transformation*. The terminology is motivated by the fact that, whilst $pq(\lambda)$ relates to $pq(-\lambda)$ via the twisting $\rho_V pq(\lambda)\rho_V = pq(-\lambda)$, performed by ρ_V, the untwisted transformation will be given by

$(\Lambda^*)^{0,1} = P(1)^{-1}P(0)\Lambda^{0,1}$ for $P(\lambda) = g^{-1}R(\lambda) \in \Gamma(O(\wedge^2\underline{\mathbb{C}}^4))$, with $g \in \Gamma(SO(\underline{\mathbb{R}}^{5,1}))$, for some $R(\lambda) \in \Gamma(O(\wedge^2\underline{\mathbb{C}}^4))$ for which no such a relation with $R(-\lambda)$ is established.

First of all, observe that, given $w := v_0 + v_+ + v_- \neq 0$, with $v_0 \in \Gamma(S_+ \wedge S_-)$, $v_+ \in \Gamma(\wedge^2 S_+)$ and $v_- \in \Gamma(\wedge^2 S_-)$, w is null if and only if

$$(v_0, v_0) = -2(v_+, v_-). \tag{9.33}$$

The nullity of w is equivalently characterized by, at each point, either $v_- = 0 = (v_0, v_0)$ or both $v_- \neq 0$ and

$$v_+ = -\frac{1}{2}(v_0, v_0)(v_-, \overline{v_-})^{-1}\overline{v_-}. \tag{9.34}$$

Lemma 9.15. *Let $\alpha \in \mathbb{C}\backslash\{0\}$. Let $v_0 \in \Gamma(S_+ \wedge S_-)$, $v_+ \in \Gamma(\wedge^2 S_+)$ and $v_- \in \Gamma(\wedge^2 S_-)$ be such that $v_0 + v_+ + v_-$ is null. In that case, $\tau(\alpha)\langle v_0 + v_+ + v_-\rangle$ is real at a point p in M if and only if, at p, either*

$$v_+ = 0 = v_-, \ \ \langle\overline{v_0}\rangle = \langle v_0\rangle$$

or

$$|(v_0, v_0)| = 1, \ \ (v_-, \overline{v_-}) = \frac{1}{2} \mid \alpha \mid^{-2}, \ \ \overline{v_0} = -v_0.$$

Proof. Our argument is pointwise, so we work in a single fibre.

Set $w := v_0 + v_+ + v_-$. From the definition of $\tau(\alpha)$, it follows that $\langle \tau(\alpha)w, \overline{\tau(\alpha)w}\rangle$ has rank 1 if and only if either

$$v_+ = 0 = v_-, \ \ \overline{v_0} = \lambda v_0$$

or

$$v_- \neq 0, \ \ \overline{v_+} = \lambda \mid \alpha \mid^2 v_-, \ \ \overline{v_0} = \lambda v_0, \ \ \mid \lambda \mid^2 = 1,$$

for some $\lambda \in \Gamma(\underline{\mathbb{C}})$. In particular, because w is null, if v_- is non-zero, then, according to (9.34), $\tau(\alpha)\langle w\rangle$ is real if and only if

$$(v_-, \overline{v_-}) = \frac{1}{2} \mid \alpha \mid^{-2} \mid (v_0, v_0) \mid \tag{9.35}$$

and

$$\overline{v_0} = -\frac{1}{2} \mid \alpha \mid^{-2} \overline{(v_0, v_0)}(v_-, \overline{v_-})^{-1} v_0. \tag{9.36}$$

Note that, if v_- is non-zero, equation (9.35) forces (v_0, v_0) to be non-zero. On the other hand, in view of the nullity of w, if (v_0, v_0) is non-zero then so is v_-. The proof is complete by observing that, in that case, together, equations (9.35) and (9.36) force $\overline{v_0} = -(v_0, v_0)^{-1}v_0$, so that $v_0 = \mid (v_0, v_0) \mid^{-2} v_0$ and, ultimately, $\mid (v_0, v_0) \mid = 1$. $\square$

Choose a non-zero $\alpha \in \mathbb{C}\backslash S^1$ and a $d_V^{\alpha,q}$-parallel null line subbundle L^α of $\wedge^2\underline{\mathbb{C}}^4$ such that, locally, $\tau(\alpha)L^\alpha$ is never real. The existence of such a choice of L^α is established in the following lemma.

Lemma 9.16. *Let l^α be a non-zero section of $\wedge^2 S_+$. Let $L^\alpha \subset \wedge^2\underline{\mathbb{C}}^4$ be a $d_V^{\alpha,q}$-parallel null line bundle defined naturally by $d_V^{\alpha,q}$-parallel transport of l_p^α, for some point $p \in M$. Then $\tau(\alpha)L^\alpha$ is never real in some open set containing p.*

Before proceeding to the proof of the lemma, observe that, at each point, the non-reality of $\tau(\alpha)L^\alpha$ is equivalent to the real subspace $\tau(\alpha)L^\alpha + \overline{\tau(\alpha)L^\alpha}$ of $(\mathbb{R}^{5,1})^{\mathbb{C}}$ being 2-dimensional, or, equivalently, non-null,

$$\overline{\tau(\alpha)L^\alpha} \cap (\tau(\alpha)L^\alpha)^\perp = \{0\}, \tag{9.37}$$

which characterizes the non-degeneracy of $\tau(\alpha)L^\alpha \oplus \overline{\tau(\alpha)L^\alpha}$.

Proof. At the point p, L^α is spanned by l_p^α, so that, by construction, at this point, $\tau(\alpha)L^\alpha$ is not real. On the other hand, at each point, the non-reality of $\tau(\alpha)L^\alpha$ is characterized by (9.37), or, equivalently, by

$$(\tau(\alpha)l, \overline{\tau(\alpha)l}\,) \neq 0,$$

fixing $l \in \Gamma(L^\alpha)$ non-zero; showing that the non-reality of $(\tau(\alpha)L^\alpha)_x$ is an open condition on $x \in M$. □

At each point, the non-reality of $\tau(\alpha)L^\alpha$, characterized by equation (9.37), establishes a decomposition

$$\wedge^2\,\underline{\mathbb{C}}^4 = \tau(\alpha)L^\alpha \oplus (\tau(\alpha)L^\alpha \oplus \overline{\tau(\alpha)L^\alpha})^\perp \oplus \overline{\tau(\alpha)L^\alpha}. \tag{9.38}$$

For $\lambda \in \mathbb{C}\backslash\{\pm\alpha, \pm\overline{\alpha}^{\,-1}\}$ and

$$r_\alpha(\lambda) := \frac{1-\overline{\alpha}^{\,-2}}{1-\alpha^2}\,\frac{\lambda-\alpha^2}{\lambda-\overline{\alpha}^{\,-2}} \neq 0, \tag{9.39}$$

set then

$$r_{\alpha,L^\alpha}(\lambda) := I \begin{cases} r_\alpha(\lambda) & \text{on } \tau(\alpha)L^\alpha \\ 1 & \text{on } (\tau(\alpha)L^\alpha + \overline{\tau(\alpha)L^\alpha})^\perp \\ r_\alpha(\lambda)^{-1} & \text{on } \overline{\tau(\alpha)L^\alpha} \end{cases},$$

defining this way a map r_{α,L^α} into $\Gamma(O(\wedge^2\underline{\mathbb{C}}^4))$ with

$$\overline{r_{\alpha,L^\alpha}(\overline{\lambda}^{\,-1})} = r_{\alpha,L^\alpha}(\lambda),$$

for all λ, which, therefore, extends holomorphically to $\mathbb{P}^1\backslash\{\pm\alpha, \pm\overline{\alpha}^{\,-1}\}$ by setting

$$r_{\alpha,L^\alpha}(\infty) := \overline{r_{\alpha,L^\alpha}(0)}.$$

We may, alternatively and for simplicity, write $r(\lambda)$ for $r_{\alpha,L^\alpha}(\lambda)$ with $\lambda\in\mathbb{P}^1\backslash\{\pm\alpha,\pm\overline{\alpha}^{\,-1}\}$. It will be useful to observe that, in the case α^2 is real, we have

$$r(\infty)=r(0)^{-1}. \tag{9.40}$$

Once and for all, fix a choice of $\sqrt{r_\alpha(0)}$, set

$$\sqrt{\overline{r_\alpha(0)}}:=\overline{\sqrt{r_\alpha(0)}},$$

and consider the corresponding $r(0),r(\infty)\in\Gamma(\mathrm{Sl}(\underline{\mathbb{C}}^4))$, for which equation (9.9) then applies:

$$r(0)\circ j=j\circ r(\infty). \tag{9.41}$$

It follows, in particular, that, at each point, the j-stability of $r(0)S_+$ is characterized by

$$r(0)S_+=r(\infty)S_-,$$

or, equivalently, by the non-complementarity of $r(0)S_+$ and $r(\infty)S_-$ in $\underline{\mathbb{C}}^4$,

$$(\wedge^2 r(0)S_+)\cap(\wedge^2 r(\infty)S_-)^\perp\neq\{0\}. \tag{9.42}$$

Next we characterize the j-stability of $r(0)S_+$ in terms of the projections of L^α with respect to the decomposition (9.15).

Lemma 9.17. *Let $l^\alpha:=v_0+v_++v_-$, with $v_0\in\Gamma(S_+\wedge S_-)$, $v_+\in\Gamma(\wedge^2S_+)$ and $v_-\in\Gamma(\wedge^2S_-)$, be a non-zero section of L^α. Let $\pi_\oplus$ and $\pi_\perp$ denote the orthogonal projections of $\wedge^2\underline{\mathbb{C}}^4$ onto $\tau(\alpha)L^\alpha\oplus\overline{\tau(\alpha)L^\alpha}$ and $(\tau(\alpha)L^\alpha\oplus\overline{\tau(\alpha)L^\alpha})^\perp$, respectively. At each point at which l^α does not vanish, the bundle $r(0)S_+$ is j-stable if and only if*

$$v_+\neq 0\neq v_- \tag{9.43}$$

and, given $u_+\in\Gamma(\wedge^2S_+)$ not zero,

$$\mid(l^\alpha,u_+)\mid\mid(l^\alpha,\overline{u_+})\mid^{-1}=\mid r_\alpha(0)\mid^2\mid\alpha\mid^{-2}, \tag{9.44}$$

together with

$$\pi_\perp u_+\neq 0,\quad\langle\pi_\perp u_+\rangle=\langle\overline{\pi_\perp u_+}\rangle \tag{9.45}$$

and

$$(\pi_\oplus r(0)u_+,\pi_\oplus\overline{r(0)u_+})\neq(\pi_\perp r(0)u_+,\pi_\perp\overline{r(0)u_+}). \tag{9.46}$$

Proof. Our argument is pointwise, so we work in a single fibre.

Let s_+, s'_+ be a frame of S_+. Then $r(0)S_+$ is j-stable if and only if

$$r(0)(s_+ \wedge s'_+) = \lambda r(\infty)(js_+ \wedge js'_+), \tag{9.47}$$

for some $\lambda \in \Gamma(\underline{\mathbb{C}})$. Note that such a λ has necessarily unit length. Indeed,

$$r(\infty)(js_+ \wedge js'_+) = \overline{r(0)(s_+ \wedge s'_+)},$$

so that

$$r(\infty)(js_+ \wedge js'_+) = \overline{\lambda}\,\overline{r(\infty)(js_+ \wedge js'_+)} = \overline{\lambda}\,r(0)(s_+ \wedge s'_+) =\mid \lambda \mid^2 r(\infty)(js_+ \wedge js'_+).$$

Write $s_+ \wedge s'_+ = a\,\tau(\alpha)l^\alpha + b\,\overline{\tau(\alpha)l^\alpha} + \pi_\perp(s_+ \wedge s'_+)$, with $a, b \in \Gamma(\underline{\mathbb{C}})$. Then

$$r(0)(s_+ \wedge s'_+) = a\,r_\alpha(0)\tau(\alpha)l^\alpha + b\,r_\alpha(0)^{-1}\overline{\tau(\alpha)l^\alpha} + \pi_\perp(s_+ \wedge s'_+),$$

whereas

$$r(\infty)(js_+ \wedge js'_+) = \overline{b}\,\overline{r_\alpha(0)}^{\,-1}\tau(\alpha)l^\alpha + \overline{a}\,\overline{r_\alpha(0)}\,\overline{\tau(\alpha)l^\alpha} + \overline{\pi_\perp(s_+ \wedge s'_+)}.$$

Hence equation (9.47) holds if and only if

$$a\,r_\alpha(0) = \lambda\,\overline{b}\,\overline{r_\alpha(0)}^{\,-1}, \;\; b\,r_\alpha(0)^{-1} = \lambda\,\overline{a}\,\overline{r_\alpha(0)}, \;\; \pi_\perp(s_+ \wedge s'_+) = \lambda\,\overline{\pi_\perp(s_+ \wedge s'_+)};$$

or, equivalently, at each point, either

$$a = 0 = b, \quad \pi_\perp(s_+ \wedge s'_+) = \lambda\,\overline{\pi_\perp(s_+ \wedge s'_+)}, \tag{9.48}$$

or $a, b \neq 0$ and

$$a\overline{b}^{\,-1} \mid r_\alpha(0) \mid^2 = \lambda = \overline{a}^{\,-1} b \mid r_\alpha(0) \mid^{-2}, \quad \pi_\perp(s_+ \wedge s'_+) = \lambda\overline{\pi_\perp(s_+ \wedge s'_+)}.$$

Note that (9.48) contradicts the complementarity of $\wedge^2 S_+$ and $\wedge^2 S_-$. Set

$$X := (\tau(\alpha)l^\alpha, \overline{\tau(\alpha)l^\alpha}) \neq 0.$$

In view of the nullity of $s_+ \wedge s'_+$,

$$(\pi_\perp(s_+ \wedge s'_+), \pi_\perp(s_+ \wedge s'_+)) = -2abX, \tag{9.49}$$

equation $a\overline{b}^{\,-1} \mid r_\alpha(0) \mid^2 = \overline{a}^{\,-1} b \mid r_\alpha(0) \mid^{-2}$, equivalent to

$$\mid b \mid = \mid a \mid\mid r_\alpha(0) \mid^2,$$

establishes

$$2 \mid a \mid^2 \mid r_\alpha(0) \mid^2 \mid X \mid = \mid (\pi_\perp(s_+ \wedge s'_+), \pi_\perp(s_+ \wedge s'_+)) \mid$$

and, in particular, that, if, at some point, $a \neq 0$, then, at that point,

$$(\pi_\perp(s_+ \wedge s'_+), \pi_\perp(s_+ \wedge s'_+)) \neq 0.$$

It follows, in particular, that, if $\pi_\perp(s_+ \wedge s'_+) = \lambda\overline{\pi_\perp(s_+ \wedge s'_+)}$, then

$$(\pi_\perp r(0)(s_+ \wedge s'_+), \overline{\pi_\perp r(0)(s_+ \wedge s'_+)}\,) \neq 0, \tag{9.50}$$

as $\pi_\perp r(0)(s_+\wedge s'_+)=\pi_\perp(s_+\wedge s'_+)$. On the other hand, as remarked previously, $r(0)S_+$ is j-stable if and only if $(r(0)(s_+\wedge s'_+),\overline{r(0)(s_+\wedge s'_+)}\,)=0$, or, equivalently,

$$(\pi_\oplus r(0)(s_+\wedge s'_+),\pi_\oplus\overline{r(0)(s_+\wedge s'_+)}\,)=-(\pi_\perp r(0)(s_+\wedge s'_+),\pi_\perp\overline{r(0)(s_+\wedge s'_+)}\,),$$

in which case, in particular, having in consideration (9.50),

$$(\pi_\oplus r(0)(s_+\wedge s'_+),\pi_\oplus\overline{r(0)(s_+\wedge s'_+)}\,)\neq(\pi_\perp r(0)(s_+\wedge s'_+),\pi_\perp\overline{r(0)(s_+\wedge s'_+)}\,).$$

We conclude that the bundle $r(0)S_+$ is j-stable if and only if $a,b,\pi_\perp(s_+\wedge s'_+)\neq 0$ and

$$\mid b\mid=\mid a\mid\mid r_\alpha(0)\mid^2, \tag{9.51}$$

together with

$$\pi_\perp(s_+\wedge s'_+)=a\overline{b}^{\,-1}\mid r_\alpha(0)\mid^2\overline{\pi_\perp(s_+\wedge s'_+)}$$

and condition (9.46). Now write $v_+=a_+\,s_+\wedge s'_+$ and $v_-=a_-\,js_+\wedge js'_+$, with $a_+,a_-\in\Gamma(\underline{\mathbb{C}})$. Set

$$N:=(s_+\wedge s'_+,js_+\wedge js'_+)>0.$$

We have

$$a=X^{-1}(s_+\wedge s'_+,\overline{\tau(\alpha)l^\alpha})=\overline{\alpha}^{\,-1}\,\overline{a_+}\,X^{-1}N,$$

as well as

$$b=X^{-1}(s_+\wedge s'_+,\tau(\alpha)l^\alpha)=\alpha\,a_-\,X^{-1}N.$$

In particular, a (respectively, b) vanishes at some point if and only if a_+ (respectively, a_-), or, equivalently, v_+ (respectively, v_-) does. We verify, on the other hand, that equation (9.51) holds if and only if

$$\mid a_-\mid=\mid a_+\mid\mid r_\alpha(0)\mid^2\mid\alpha\mid^{-2},$$

or, equivalently, if condition (9.44) is satisfied. It follows, in particular, that, together, conditions (9.43) and (9.44) ensure that

$$\pi_\oplus r(0)(s_+\wedge s'_+)=\lambda\,\pi_\oplus r(\infty)(js_+\wedge js'_+),$$

for $\lambda:=a\overline{b}^{\,-1}\mid r_\alpha(0)\mid^2\in S^1$, which together with condition (9.45), and given that

$$\pi_\perp r(0)(s_+\wedge s'_+)=\pi_\perp(s_+\wedge s'_+)=\overline{\pi_\perp r(\infty)(js_+\wedge js'_+)},$$

amounts to

$$r(0)(s_+\wedge s'_+)=\lambda\,\pi_\oplus r(\infty)(js_+\wedge js'_+)+\xi\,\pi_\perp r(\infty)(js_+\wedge js'_+),$$

for some $\xi\in\Gamma(\underline{\mathbb{C}})$. In fact, we verify that ξ is unit:

$$\mid(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+))\mid=\mid\xi\mid^2\mid(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+))\mid,$$

and condition (9.43) ensures, on the other hand, according to (9.49), that

$$(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+))\neq 0. \tag{9.52}$$

For simplicity, write $\pi^r_\oplus$ for $\pi_\oplus r(\infty)(js_+\wedge js'_+)$. By the nullity of $s_+\wedge s'_+$, it follows that

$$0=(r(0)(s_+\wedge s'_+),r(0)(s_+\wedge s'_+))=\lambda^2(\pi^r_\oplus,\pi^r_\oplus)+\xi^2(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+)),$$

as well as

$$0=(r(\infty)(js_+\wedge js'_+)),r(\infty)(js_+\wedge js'_+)))=(\pi^r_\oplus,\pi^r_\oplus)+(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+)).$$

Hence

$$(\pi^r_\oplus,\pi^r_\oplus)=-(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+)) \tag{9.53}$$

and, therefore, $\xi^2=\lambda^2$, which, as ξ and λ are both unit, forces $\lambda,\xi\in\{-1,1,-i,i\}$. Together with (9.52) and (9.53), the nullity of $r(0)(s_+\wedge s'_+)$ excludes the cases $\xi=\pm i\lambda$. Indeed, if $r(0)(s_+\wedge s'_+)=\lambda\,\pi^r_\oplus\pm i\lambda\,\pi_\perp(s_+\wedge s'_+)$, then

$$0=(\lambda\,\pi^r_\oplus\pm i\lambda\,\pi_\perp(s_+\wedge s'_+),\lambda\,\pi^r_\oplus\pm i\lambda\,\pi_\perp(s_+\wedge s'_+)),$$

respectively, leading to a contradiction:

$$0=\lambda^2(\pi^r_\oplus,\pi^r_\oplus)-\lambda^2(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+))=-2\lambda^2(\pi_\perp(s_+\wedge s'_+),\pi_\perp(s_+\wedge s'_+))\neq 0.$$

Together with condition (9.46), and given that $r(0)(s_+\wedge s'_+)$ and $r(\infty)(js_+\wedge s'_+)$ are complex conjugate of each other, the nullity of $r(0)(s_+\wedge s'_+)$ excludes, on the other hand, the case $\xi=-\lambda$: if $r(0)(s_+\wedge s'_+)=\lambda\,\pi^r_\oplus-\lambda\,\pi_\perp(s_+\wedge s'_+)$, then

$$0=(\lambda\,\pi^r_\oplus-\lambda\,\pi_\perp(s_+\wedge s'_+),\overline{\pi^r_\oplus+\pi_\perp(s_+\wedge s'_+)}\,)=\lambda(\pi^r_\oplus,\overline{\pi^r_\oplus})-\lambda(\pi_\perp(s_+\wedge s'_+),\overline{\pi_\perp(s_+\wedge s'_+)}\,)$$

and, therefore, $(\pi^r_\oplus,\overline{\pi^r_\oplus})=(\pi_\perp(s_+\wedge s'_+),\overline{\pi_\perp(s_+\wedge s'_+)}\,)$, or, equivalently,

$$(\pi_\oplus\overline{r(0)(s_+\wedge s'_+)},\pi_\oplus r(0)(s_+\wedge s'_+))=(\pi_\perp r(0)(s_+\wedge s'_+),\pi_\perp\overline{r(0)(s_+\wedge s'_+)}\,).$$

We conclude that $\xi=\lambda$, completing the proof. □

At each point, the non-j-stability of the bundle $r(0)S_+$, of 2-planes in $\mathbb{C}^4$, is equivalent to its complementarity to $jr(0)S_+=r(\infty)S_-$ in $\underline{\mathbb{C}}^4$. Choose L^α for which, furthermore, locally, $r(0)S_+$ and $r(\infty)S_-$ are complementary in $\underline{\mathbb{C}}^4$. The existence of such a choice is established in the following lemma.

Lemma 9.18. *Let l^α be a non-zero section of $\wedge^2S_+$. Let $L^\alpha\subset\wedge^2\underline{\mathbb{C}}^4$ be a $d^{\alpha,q}_V$-parallel null line bundle defined naturally by $d^{\alpha,q}_V$-parallel transport of l^α_p, for some point $p\in M$. Then there is some open set containing p in which $\tau(\alpha)L^\alpha$ is never real and $r(0)S_+$ and $r(\infty)S_-$ are complementary in $\underline{\mathbb{C}}^4$.*

Proof. By construction,

$$L^\alpha_p=\langle l^\alpha_p\rangle,$$

so that, at the point p, $\tau(\alpha)L^\alpha$ is not real and, therefore, r is, indeed, defined; whilst Lemma 9.17 ensures that, at p, $r(0)S_+\cap r(\infty)S_-=\{0\}$. On the other hand, according

to (9.42), at each point, the non-j-stability of $r(0)S_+$ is characterized by

$$(r(0)\wedge^2 S_+)\cap(\overline{r(0)\wedge^2 S_+})^\perp=\{0\},$$

or, equivalently, by $(r(0)\,u_+,\overline{r(0)\,u_+})\neq 0$, fixing $u_+\in\Gamma(\wedge^2 S_+)$ non-zero; which shows that the non-j-stability of $(r(0)S_+)_x$ is an open condition on $x\in M$. And so is the non-reality of $(\tau(\alpha)L^\alpha)_x$, as remarked previously. □

Set

$$S_+^* := r(0)S_+,\;\; S_-^* := jS_+^* = r(\infty)S_-$$

and, considering projections $\pi_{S_+^*}:\underline{\mathbb{C}}^4\to S_+^*$ and $\pi_{S_-^*}:\underline{\mathbb{C}}^4\to S_-^*$ with respect to the decomposition

$$\underline{\mathbb{C}}^4 = S_+^*\oplus S_-^*,$$

define two line bundles by

$$L_+^* := \pi_{S_+^*}r(\infty)L_+,\quad L_-^* := jL_+^* = \pi_{S_-^*}r(0)L_-. \tag{9.54}$$

As we know, L^α is a $d_V^{\alpha,q}$-parallel null line bundle if and only if $\rho_V L^\alpha$ is a $d_V^{-\alpha,q}$-parallel null line bundle. Note that the reality of $\tau(\alpha)L^\alpha$ is equivalent to that of $\tau(-\alpha)\rho_V L^\alpha$, and that

$$r_{\alpha,L^\alpha} = r_{-\alpha,\rho_V L^\alpha}.$$

We define the *untwisted Bäcklund transform L^* of L of parameters α, L^α, or, equally,* $-\alpha,\rho_V L^\alpha$, by setting

$$\wedge^2 L^* := L_+^*\wedge L_-^*.$$

The real line bundle $\wedge^2 L^*$ determines a j-stable bundle L^* of 2-planes in $\underline{\mathbb{C}}^4$.

Theorem 9.19. *L^* is a constrained Willmore surface in S^4, provided that it immerses.*

The proof of the theorem will follow some preliminaries, presented next.

At each point, the complementarity of S_+^* and S_-^* in $\underline{\mathbb{C}}^4$ establishes a decomposition

$$\wedge^2\underline{\mathbb{C}}^4 = \wedge^2 S_+^*\oplus S_+^*\wedge S_-^*\oplus\wedge^2 S_-^*.$$

Define then, for $\lambda\in\mathbb{C}\backslash\{0\}$, another orthogonal transformation of $\wedge^2\underline{\mathbb{C}}^4$ by

$$\tau^*(\lambda) := I\begin{cases}\lambda^{-1} & \text{on } \wedge^2 S_+^*\\ 1 & \text{on } S_+^*\wedge S_-^*\\ \lambda & \text{on } \wedge^2 S_-^*\end{cases}.$$

Set

$$V^* := S_+^*\wedge S_-^* \tag{9.55}$$

and let ρ_{V^*} denote reflection across V^*. Observe that $\tau(-1)=\rho_V$ and $\tau^*(-1)=\rho_{V^*}$, and that, for $\lambda\in\mathbb{C}\backslash\{0\}$,

$$\tau(-\lambda)=\tau(\lambda)\rho_V,\quad \tau^*(-\lambda)=\tau^*(\lambda)\rho_{V^*} \tag{9.56}$$

and

$$\overline{\tau(\lambda)} = \tau(\overline{\lambda}^{\,-1}), \quad \overline{\tau^*(\lambda)} = \tau^*(\overline{\lambda}^{\,-1}). \tag{9.57}$$

For $\lambda \in \mathbb{C}\backslash\{0\}$, set

$$R(\lambda) := \tau^*(\lambda)^{-1}\, r_{\alpha,L^\alpha}(\lambda^2)\, \tau(\lambda) \in \Gamma(O(\wedge^2\underline{\mathbb{C}}^4)),$$

which will play a crucial role on what follows. Fix $g \in \Gamma(SO(\underline{\mathbb{R}}^{5,1}))$ such that

$$g\rho_V g^{-1} = \rho_{V^*}$$

and set also

$$P(\lambda) := g^{-1}R(\lambda).$$

The holomorphicity of $P(\lambda)$ at $\lambda = 0$, equivalent to that of $R(\lambda)$, is directly ensured by Lemma 6.28, and, similarly, after an appropriate change of variable, so is the holomorphicity of $P(\lambda)$ at $\lambda = \infty$. Set then

$$(\Lambda^*)^{1,0} := P(1)^{-1}P(\infty)\,\Lambda^{1,0}$$

and

$$(\Lambda^*)^{0,1} := P(1)^{-1}P(0)\,\Lambda^{0,1}.$$

Equivalently,

$$(\Lambda^*)^{1,0} = R(\infty)\,(L_+ \wedge S_-), \quad (\Lambda^*)^{0,1} = R(0)\,(L_- \wedge S_+),$$

as $R(1) = I$. Observe that $(\Lambda^*)^{1,0}$ and $(\Lambda^*)^{0,1}$ are complex conjugate of each other: having in consideration (9.57),

$$\begin{array}{rcl} \overline{(\Lambda^*)^{1,0}} & = & \lim_{\lambda\to\infty} \tau^*(\overline{\lambda})\,\overline{r(\lambda^2)}\,\tau(\overline{\lambda}^{\,-1})\,\overline{L_+ \wedge S_-} \\ & = & \lim_{\lambda\to 0} \tau^*(\lambda^{-1})\,r(0)\,\tau(\lambda)\,(L_- \wedge S_+) \\ & = & (\Lambda^*)^{0,1}. \end{array}$$

Fixing a choice of $\sqrt{\lambda}$,

$$\tau(\lambda) = I \begin{cases} \sqrt{\lambda}^{\,-1} & \text{on } S_+ \\ \sqrt{\lambda} & \text{on } S_- \end{cases}$$

and

$$\tau^*(\lambda) = I \begin{cases} \sqrt{\lambda}^{\,-1} & \text{on } S_+^* \\ \sqrt{\lambda} & \text{on } S_-^* \end{cases},$$

and, therefore, given $s_+ \in \Gamma(S_+)$, $R(0)s_+ = \lim_{\lambda\to 0}(\pi_{S_+^*}\, r(\lambda^2)\, s_+ + \lambda^{-1}\pi_{S_-^*}\, r(\lambda^2)\, s_+)$. On the other hand, as $r(\lambda^2)$ is holomorphic at $\lambda = 0$, it admits a Taylor expansion

$$r(\lambda^2) = r(0) + \lambda^2 \frac{d}{d\lambda}_{|_{\lambda=0}} r(\lambda^2) + \frac{\lambda^4}{2}\frac{d^2}{d\lambda^2}_{|_{\lambda=0}} r(\lambda^2) + \ldots.$$

around 0, making clear that

$$\lim_{\lambda\to 0}\lambda^{-1}r(\lambda^2)=\lim_{\lambda\to 0}\lambda^{-1}r(0). \tag{9.58}$$

Hence $R(0)s_+ = r(0)s_+ + \pi_{S_-^*}(\lim_{\lambda\to 0}\lambda^{-1}\,r(0)\,s_+)$. Now the holomorphicity of $R(\lambda)$ at $\lambda=0$ forces $\pi_{S_-^*}\,r(0)s_+$ to be 0, establishing $R(0)s_+=r(0)s_+$. Thus $R(0)S_+=S_+^*$. A similar, even slightly simpler computation establishes $R(0)L_- = \pi_{S_-^*}r(0)L_-$. We conclude that

$$(\Lambda^*)^{0,1}=\pi_{S_-^*}r(0)L_-\wedge S_+^*, \tag{9.59}$$

or, equivalently, by complex conjugation,

$$(\Lambda^*)^{1,0}=\pi_{S_+^*}r(\infty)L_+\wedge S_-^*.$$

Thus

$$(\Lambda^*)^{1,0}\cap(\Lambda^*)^{0,1}=\wedge^2L^*=:\Lambda^*.$$

The choice of the notation Λ^*, already attributed to a twisted Bäcklund transform of Λ, is not casual, as we shall verify in the next section.

Now we proceed to the proof of Theorem 9.19.

Proof. Note that, according to (9.56),

$$P(-\lambda)=g^{-1}\tau^*(-\lambda)\,r_{\alpha,L^\alpha}(\lambda^2)\,\tau(-\lambda)=g^{-1}\rho_{V^*}R(\lambda)\rho_V=\rho_V P(\lambda)\rho_V. \tag{9.60}$$

The proof will consist of showing that

$$\lambda\mapsto d_P^{\lambda,q}:=P(\lambda)\circ d_V^{\lambda,q}\circ P(\lambda)^{-1}$$

admits a holomorphic extension to $\lambda\in\mathbb{C}\backslash\{0\}$ through metric connections on $\wedge^2\underline{\mathbb{C}}^4$. It will then follow from Theorem 6.25 that Λ^* is a constrained Willmore surface, provided that it immerses, with no need for condition (6.23) to be verified, as it would solely intend to ensure that $(\Lambda^*)^{1,0}$ and $(\Lambda^*)^{0,1}$ intersect in a rank 1 bundle, a fact that is already known to us.

Since $d^V_{\alpha^2,q}$ is metric,

$$d^V_{\alpha^2,q}\,\Gamma(\,\overline{\tau(\alpha)L^\alpha}\,)\subset\Omega^1(\,\overline{\tau(\alpha)L^\alpha}^{\,\perp}),$$

as well as, in view of the parallelness of $\tau(\alpha)L^\alpha$ with respect to $d^V_{\alpha^2,q}$,

$$d^V_{\alpha^2,q}\,\Gamma((\tau(\alpha)L^\alpha\oplus\overline{\tau(\alpha)L^\alpha})^\perp)\subset\Omega^1((\tau(\alpha)L^\alpha)^\perp).$$

Let $\pi_{\tau L}:\wedge^2\underline{\mathbb{C}}^4\to\tau(\alpha)L^\alpha$ and $\pi_{\overline{\tau L}}:\wedge^2\underline{\mathbb{C}}^4\to\overline{\tau(\alpha)L^\alpha}$ be projections with respect to the decomposition (9.38). Let $\pi_\oplus$ and $\pi_\perp$ be as in Lemma 9.17. As

$$(\tau(\alpha)L^\alpha)^\perp=\tau(\alpha)L^\alpha\oplus(\tau(\alpha)L^\alpha\oplus\overline{\tau(\alpha)L^\alpha})^\perp$$

and

$$\overline{\tau(\alpha)L^{\alpha}}^{\,\perp} = \overline{\tau(\alpha)L^{\alpha}} \oplus (\tau(\alpha)L^{\alpha} \oplus \overline{\tau(\alpha)L^{\alpha}})^{\perp},$$

we conclude that $\pi_{\tau L} \circ d^V_{\alpha^2,q} \circ \pi_{\overline{\tau L}} = 0 = \pi_{\overline{\tau L}} \circ d^V_{\alpha^2,q} \circ \pi_{\perp}$, showing that $d^V_{\alpha^2,q}$ splits as

$$d^V_{\alpha^2,q} = D^q_{\alpha^2} + \beta^q_{\alpha^2},$$

for the connection

$$D^q_{\alpha^2} := d^V_{\alpha^2,q} \circ \pi_{\tau L} + \pi_{\overline{\tau L}} \circ d^V_{\alpha^2,q} \circ \pi_{\overline{\tau L}} + \pi_{\perp} \circ d^V_{\alpha^2,q} \circ \pi_{\perp},$$

on $\wedge^2\underline{\mathbb{C}}^4$, and

$$\beta^q_{\alpha^2} := \pi_{\perp} \circ d^V_{\alpha^2,q} \circ \pi_{\overline{\tau L}} + \pi_{\tau L} \circ d^V_{\alpha^2,q} \circ \pi_{\perp} \in \Omega^1(\tau(\alpha)L^{\alpha} \wedge (\tau(\alpha)L^{\alpha} \oplus \overline{\tau(\alpha)L^{\alpha}}\,)^{\perp}).$$

For simplicity, denote $r_{\alpha}(\lambda^2) \in \mathbb{C}$ (as defined in (9.39)) by α_{λ^2}. Clearly, for each λ,

$$r(\lambda^2) \circ D^q_{\alpha^2} \circ r(\lambda^2)^{-1} = D^q_{\alpha^2}, \quad r(\lambda^2)\,\beta^q_{\alpha^2}\,r(\lambda^2)^{-1} = \alpha_{\lambda^2}\,\beta^q_{\alpha^2}.$$

Now decompose $d^V_{\lambda^2,q}$ as

$$d^V_{\lambda^2,q} = d^V_{\alpha^2,q} + (\lambda^2 - \alpha^2)A(\lambda^2),$$

for $\lambda \in \mathbb{C}\backslash\{0, \pm\alpha\}$, with $\lambda \mapsto A(\lambda^2) \in \Omega^1(o(\wedge^2\underline{\mathbb{C}}^4))$ holomorphic. It follows that

$$\begin{aligned} d^{\lambda,q}_P &= g^{-1}\tau^*(\lambda)^{-1} r_{\alpha,L^{\alpha}}(\lambda^2) \circ d^V_{\lambda^2,q} \circ r_{\alpha,L^{\alpha}}(\lambda^2)^{-1}\tau^*(\lambda) g \\ &= g^{-1}\tau^*(\lambda)^{-1} D^q_{\alpha^2}\,\tau^*(\lambda)\,g + \alpha_{\lambda^2}\,g^{-1}\tau^*(\lambda)^{-1}\beta^q_{\alpha^2}\,\tau^*(\lambda)\,g + \Psi(\lambda), \end{aligned}$$

for $\Psi(\lambda) := g^{-1}\tau^*(\lambda)^{-1} r(\lambda^2)(\lambda^2 - \alpha^2)A(\lambda^2) r(\lambda^2)^{-1}\tau^*(\lambda)\,g$ and $\lambda \in \mathbb{C}\backslash\{0, \pm\alpha, \pm\overline{\alpha}^{\,-1}\}$. Set $\Upsilon(\lambda) := (\lambda^2 - \alpha^2) r(\lambda^2) A(\lambda^2) r(\lambda^2)^{-1}$. The skew-symmetry of $A(\lambda^2)$ establishes

$$A(\lambda^2)\tau(\alpha)L^{\alpha} \subset (\tau(\alpha)L^{\alpha})^{\perp}, \quad A(\lambda^2)\overline{\tau(\alpha)L^{\alpha}} \subset \overline{\tau(\alpha)L^{\alpha}}^{\,\perp}$$

and, consequently,

$$\pi_{\overline{\tau L}}\,A(\lambda^2)\,\pi_{\tau L} = 0 = \pi_{\tau L}\,A(\lambda^2)\,\pi_{\overline{\tau L}}.$$

On the other hand, it is clear that

$$\pi_{\overline{\tau L}}\,r(\lambda^2)\,A(\lambda^2)\,r(\lambda^2)^{-1}\pi_{\tau L} = a^{-1}_{\lambda^2}\,\pi_{\overline{\tau L}}\,A(\lambda^2)\,a^{-1}_{\lambda^2}\,\pi_{\tau L} = a^{-2}_{\lambda^2}\,\pi_{\overline{\tau L}}\,A(\lambda^2)\pi_{\tau L}$$

and, similarly,

$$\pi_{\tau L}\,r(\lambda^2)\,A(\lambda^2)\,r(\lambda^2)^{-1}\pi_{\overline{\tau L}} = a^2_{\lambda^2}\,\pi_{\tau L}\,A(\lambda^2)\pi_{\overline{\tau L}}.$$

Hence $\pi_{\overline{\tau L}}\,\Upsilon(\lambda)\,\pi_{\tau L} = 0 = \pi_{\tau L}\Upsilon(\lambda)\,\pi_{\overline{\tau L}}$. It follows that

$$\begin{aligned} \Upsilon(\lambda) &= (\lambda^2 - \alpha^2)\,(\pi_{\tau L}\,A(\lambda^2)\,\pi_{\tau L} + \pi_{\overline{\tau L}}\,A(\lambda^2)\,\pi_{\overline{\tau L}} + \pi_{\perp}A(\lambda^2)\,\pi_{\perp}) \\ &\quad + \frac{1-\alpha^2}{1-\overline{\alpha}^{\,-2}}\,(\lambda^2 - \overline{\alpha}^{\,-2})\,(\pi_{\perp}\,A(\lambda^2)\,\pi_{\tau L} + \pi_{\overline{\tau L}}\,A(\lambda^2)\,\pi_{\perp}) \\ &\quad + \frac{1-\overline{\alpha}^{\,-2}}{1-\alpha^2}\,\frac{(\lambda^2-\alpha^2)^2}{\lambda^2 - \overline{\alpha}^{\,-2}}\,(\pi_{\perp}\,A(\lambda^2)\,\pi_{\overline{\tau L}} + \pi_{\tau L}\,A(\lambda^2)\,\pi_{\perp}). \end{aligned}$$

Hence, by setting

$$\begin{aligned} d_P^{\alpha,q} \;:=\;& g^{-1}\tau^*(\alpha)^{-1}D^q_{\alpha^2}\,\tau^*(\alpha)\,g \\ &+ g^{-1}\tau^*(\alpha)^{-1}\frac{1-\alpha^2}{1-\overline{\alpha}^{\,-2}}\,(\alpha^2-\overline{\alpha}^{\,-2})\,(\pi_\perp\,A(\alpha^2)\,\pi_{\tau L}+\pi_{\overline{\tau L}}\,A(\alpha^2)\,\pi_\perp)\tau^*(\alpha)g \end{aligned}$$

and

$$\begin{aligned} d_P^{-\alpha,q} \;:=\;& g^{-1}\tau^*(-\alpha)^{-1}D^q_{\alpha^2}\,\tau^*(-\alpha)\,g \\ &+ g^{-1}\tau^*(-\alpha)^{-1}\frac{1-\alpha^2}{1-\overline{\alpha}^{\,-2}}\,(\alpha^2-\overline{\alpha}^{\,-2})\,(\pi_\perp\,A(\alpha^2)\,\pi_{\tau L}+\pi_{\overline{\tau L}}\,A(\alpha^2)\,\pi_\perp)\tau^*(-\alpha)g, \end{aligned}$$

we extend holomorphically $(\lambda \mapsto d_P^{\lambda,q})$ to $\lambda \in \mathbb{C}\backslash\{0,\pm\overline{\alpha}^{\,-1}\}$ through what, by continuity, are metric connections on $\wedge^2\underline{\mathbb{C}}^4$.

The existence of a holomorphic extension to $\mathbb{C}\backslash\{0\}$, through metric connections on $\wedge^2\underline{\mathbb{C}}^4$, can be proved analogously, having in consideration the parallelness of $\overline{\tau(\alpha)L^\alpha}$ with respect to the connection

$$d^V_{\overline{\alpha}^{\,-2},q} = \overline{\tau(\alpha)\circ d_V^{\alpha,q}\circ\tau(\alpha)^{-1}}.$$

□

Suppose L^* immerses. Note that $P(1)^{-1}P(-1) = R(-1) = \rho_{V^*}\rho_V$, whilst, on the other hand, according to (9.60), $P(1)^{-1}P(-1) = P(1)^{-1}\rho_V P(1)\rho_V$. We conclude that $\rho_{V^*}P(1)^{-1} = P(1)^{-1}\rho_V$ and, therefore, as $\rho_V V = V$, that $\rho_{V^*}P(1)^{-1}V = P(1)^{-1}V$. Equivalently,

$$V^* = P(1)^{-1}V,$$

V^* is the complexification of the central sphere congruence of Λ^*. Denote the mean curvature sphere of L^* by S^*. In view of (9.55), the eigenspace of S^* associated to the eigenvalue i is either S^*_+ or S^*_-. Suppose it is S^*_-. In that case, $(\Lambda^*)^{0,1} = (L^*\cap S^*_+)\wedge S^*_-$, which, in view of the complementarity of S^*_+ and S^*_- in $\underline{\mathbb{C}}^4$, contradicts (9.59). Hence S^*_+ is the eigenspace of S^* associated to i. The choice of notation is consistent. It follows, in particular, that the equalities (9.54) are not merely formal either.

9.3.2. Twisted vs. untwisted Bäcklund transformation of constrained Willmore surfaces in 4-space. Twisted Bäcklund transformation of constrained Willmore surfaces in 4-space is closely related to untwisted Bäcklund transformation. As we verify in this section, when twisted Bäcklund transformation parameters constitute untwisted Bäcklund transformation parameters, the corresponding twisted and untwisted Bäcklund transforms coincide.[2]

[2]In Appendix B below, we verify that twisted and untwisted Bäcklund transformation parameters conditions at a point are not equivalent.

Choose a non-zero $\alpha \in \mathbb{C}\backslash S^1$ and a $d_V^{\alpha,q}$-parallel null line subbundle L^α of $\wedge^2\underline{\mathbb{C}}^4$ such that, locally, $\tau(\alpha)L^\alpha$ is never real, $r(0)S_+$ and $r(\infty)S_-$ are complementary in $\underline{\mathbb{C}}^4$, and condition (9.31) is satisfied. The existence of such a choice of L^α is established in the next lemma. First observe that, according to (9.33), given $w := v_0 + v_+ + v_-$ null, with $v_0 \in \Gamma(S_+ \wedge S_-)$, $v_+ \in \Gamma(\wedge^2 S_+)$ and $v_- \in \Gamma(\wedge^2 S_-)$, $\rho_V w$ is orthogonal to w, at some point, if and only if, at that point, $(v_0, v_0) = 0$, or, equivalently, as $\wedge^2 S_+ \cap (\wedge^2 S_-)^\perp = \{0\}$, either $v_+ = 0$ or $v_- = 0$.

Lemma 9.20. *Let v_- be a non-zero section of $\wedge^2 S_-$ with $(v_-, \overline{v_-}) \neq 1$. Let v_0 be a section of $S_+ \wedge S_-$ with $(v_0, v_0) = (v_-, \overline{v_-})$ and $(v_0, \overline{v_0}) = \frac{1}{4}(v_-, \overline{v_-})$. Define a null section of $\wedge^2\underline{\mathbb{C}}^4$ by $l^\alpha := v_0 - \frac{1}{2}\overline{v_-} + v_-$. Let $L^\alpha \subset \wedge^2\underline{\mathbb{C}}^4$ be a $d_V^{\alpha,q}$-parallel null line bundle defined naturally by $d_V^{\alpha,q}$-parallel transport of l_p^α, for some point $p \in M$. Then there is a (non-empty) open set where L^α is never orthogonal to $\rho_V L^\alpha$, $\tilde{L}^\alpha$ is never orthogonal to $\rho_V \tilde{L}^\alpha$, $\tau(\alpha)L^\alpha$ is never real and $r(0)S_+$ and $r(\infty)S_-$ are complementary in $\underline{\mathbb{C}}^4$.*

PROOF. At the point p, L^α is spanned by l_p^α. The fact that, at the point p, in particular, (v_0, v_0) is not zero establishes the non-orthogonality of L^α and $\rho_V L^\alpha$ at this point. The non-reality of $\tau(\alpha)L^\alpha$ at p follows then, according to Lemma 9.15, from the fact that $\mid (v_0, v_0) \mid = (v_-, \overline{v_-}) \neq 1$. Let $\pi_\perp$ denote the orthogonal projection of $\wedge^2\underline{\mathbb{C}}^4$ onto $(\tau(\alpha)L^\alpha \oplus \overline{\tau(\alpha)L^\alpha})^\perp$. The non-$j$-stability of $r(0)S_+$ at the point p is established, according to Lemma 9.17, by the fact that $\langle \pi_\perp v_- \rangle \neq \langle \pi_\perp \overline{v_-} \rangle$, as we verify next. First of all, note that $\tau(\alpha)l^\alpha = -\frac{1}{2}\alpha^{-1}\overline{v_-} + v_0 + \alpha v_-$. Set $X := (\tau(\alpha)l^\alpha, \overline{\tau(\alpha)l^\alpha}) \neq 0$. Then $\overline{v_-} = X^{-1}(\overline{v_-}, \overline{\tau(\alpha)l^\alpha})\tau(\alpha)l^\alpha + X^{-1}(\overline{v_-}, \tau(\alpha)l^\alpha)\overline{\tau(\alpha)l^\alpha} + \pi_\perp \overline{v_-}$, or, equivalently,

$$\begin{aligned}(v_-, \overline{v_-})^{-1} X\, \overline{v_-} &= \alpha\overline{\alpha}^{-1} v_- + (\frac{1}{4} \mid \alpha \mid^{-2} + \mid \alpha \mid^2)\overline{v_-} \\ &\quad - \frac{1}{2}\,\overline{\alpha}^{-1} v_0 + \alpha\,\overline{v_0} + (v_-, \overline{v_-})^{-1} X \pi_\perp \overline{v_-},\end{aligned}$$

and, therefore,

$$\begin{aligned}(v_-, \overline{v_-})^{-1} X\, \pi_\perp \overline{v_-} &= -\alpha\overline{\alpha}^{-1} v_- + \frac{1}{2}\,\overline{\alpha}^{-1} v_0 - \alpha\,\overline{v_0} \\ &\quad - (\frac{1}{4} \mid \alpha \mid^{-2} + \mid \alpha \mid^2 - (v_-, \overline{v_-})^{-1} X)\overline{v_-}.\end{aligned}$$

Hence the linear dependency of $\pi_\perp \overline{v_-}$ and its complex conjugate implies, in particular,

$$\alpha\overline{\alpha}^{-1} = \overline{\alpha}\alpha^{-1} \tag{9.61}$$

and

$$(\frac{1}{2}\,\alpha^{-1} + \alpha)\overline{v_0} = (\frac{1}{2}\,\overline{\alpha}^{-1} - \overline{\alpha})v_0. \tag{9.62}$$

Condition (9.61) consists of the reality of α^2, whereas equation (9.62) forces, in particular,

$$v_0 = \frac{(1-2\overline{\alpha}^2)(1-2\alpha^2)}{(1+2\alpha^2)(1+2\overline{\alpha}^2)}\, v_0.$$

As v_0 is non-zero, we conclude that, if $\langle \pi_\perp \overline{v_-}, \pi_\perp v_- \rangle$ has rank 1, then $\frac{1-2\alpha^2}{1+2\alpha^2} = \pm 1$, which contradicts the fact that α is non-zero.

Now consider projections $\pi_{\overline{L^\alpha}} : \wedge^2 \underline{\mathbb{C}}^4 \to \overline{L^\alpha}$, $\pi_{\rho_V \overline{L^\alpha}} : \wedge^2 \underline{\mathbb{C}}^4 \to \rho_V \overline{L^\alpha}$ and $\pi_{\perp'} : \wedge^2 \underline{\mathbb{C}}^4 \to (\overline{L^\alpha} \oplus \rho_V \overline{L^\alpha})^\perp$ with respect to the decomposition

$$\wedge^2 \underline{\mathbb{C}}^4 = \overline{L^\alpha} \oplus (\overline{L^\alpha} \oplus \rho_V \overline{L^\alpha})^\perp \oplus \rho_V \overline{L^\alpha},$$

provided by the non-orthogonality of $\overline{L^\alpha}$ and $\rho_V \overline{L^\alpha}$, consequent to the non-orthogonality of L^α and $\rho_V L^\alpha$. Set

$$A := \frac{\alpha - \overline{\alpha}^{-1}}{\alpha + \overline{\alpha}^{-1}} = \frac{\mid \alpha \mid^2 - 1}{\mid \alpha \mid^2 + 1} \in \mathbb{R}.$$

Then

$$\rho_V q(\alpha) l^\alpha = A\, \rho_V \pi_{\overline{L^\alpha}}\, l^\alpha + \rho_V \pi_{\perp'}\, l^\alpha + A^{-1} \rho_V \pi_{\rho_V \overline{L^\alpha}}\, l^\alpha,$$

and, therefore,

$$\begin{aligned}(\rho_V q(\alpha) l^\alpha, q(\alpha) l^\alpha) \quad = \quad & A^2 (\pi_{\overline{L^\alpha}}\, l^\alpha, \rho_V \pi_{\overline{L^\alpha}}\, l^\alpha) \\ & + (\pi_{\perp'} l^\alpha, \rho_V \pi_{\perp'}\, l^\alpha) \\ & + A^{-2} (\pi_{\rho_V \overline{L^\alpha}}\, l^\alpha, \rho_V \pi_{\rho_V \overline{L^\alpha}}\, l^\alpha).\end{aligned}$$

At the point p,

$$\pi_{\overline{L^\alpha}}\, l^\alpha = a(\overline{v_0} - \frac{1}{2} v_- + \overline{v_-}), \qquad \pi_{\rho_V \overline{L^\alpha}}\, l^\alpha = b(\overline{v_0} + \frac{1}{2} v_- - \overline{v_-}),$$

for some $a, b \in \mathbb{C}$, and, therefore,

$$\pi_{\perp'} l^\alpha = v_0 - (a+b)\overline{v_0} - (\frac{1}{2} + a - b)\overline{v_-} + (1 + \frac{1}{2}(a-b)) v_-.$$

The orthogonality relations $(\pi_{\perp'} l^\alpha, \overline{l^\alpha}) = 0 = (\pi_{\perp^\alpha} l^\alpha, \rho_V \overline{l^\alpha})$ establish then $a = \frac{-1}{2}$ and $b = \frac{3}{4}$. It follows that, at the point p,

$$(v_-, \overline{v_-})^{-1} (\rho_V q(\alpha) l^\alpha, q(\alpha) l^\alpha) = \frac{1}{2} A^2 + \frac{3}{8} + \frac{9}{8} A^{-2}.$$

Ultimately, the fact that A is real, and, therefore, A^2 and A^{-2} are positive, shows that, at p,

$$(\rho_V q(\alpha) l^\alpha, q(\alpha) l^\alpha) \neq 0.$$

The proof is complete by observing that the non-orthogonality of $\rho_V L^\alpha$ and L^α, characterized by $(\rho_V l, l) \neq 0$, fixing $l \in \Gamma(L^\alpha)$ non-zero, is an open condition on the points in M. And so is, similarly, the non-orthogonality of $\rho_V \tilde{L}^\alpha$ and $\tilde{L}^\alpha$. And so are the non-reality of $\tau(\alpha) L^\alpha$ and the non-j-stability of $r(0) S_+$, as observed previously. $\square$

For such a choice of parameters, we are in a position to refer to both $R(\lambda)$, as defined in the previous section, and $(pq)^{-1}(1)pq(\lambda)$.

Proposition 9.21.

$$R(\lambda) = (pq)^{-1}(1)(pq)(\lambda), \tag{9.63}$$

for all λ.

The proof of the proposition is based on Lemma 6.28.

Proof. Equation (9.63) holds for $\lambda = 1$. The proof will consist of showing that $\xi := \tau^*(\lambda)^{-1}\, r(\lambda^2)\, \tau(\lambda) q^{-1}(\lambda) p^{-1}(\lambda)(pq)(1)$ is holomorphic in $\mathbb{P}^1$ (and, therefore, constant). For that, first note that, at most, ξ has simple poles at $0, \infty, \pm\alpha$ and $\pm\overline{\alpha}^{-1}$. The holomorphicity of ξ at 0, equivalent to the holomorphicity of $\tau^*(\lambda)^{-1}\, r(\lambda^2)\, \tau(\lambda)$ at $\lambda = 0$, is already known to us, from the previous section, as well as the holomorphicity of ξ at $\lambda = \infty$. Observe, on the other hand, that we can decompose $r(\lambda^2)$ as

$$r(\lambda^2) = r_1(\lambda) r_2(\lambda) = r_2(\lambda) r_1(\lambda),$$

for

$$r_1(\lambda) := I \begin{cases} c\, \frac{\lambda - \alpha}{\lambda - \overline{\alpha}^{-1}} & \text{on}\, \tau(\alpha) L^\alpha \\ 1 & \text{on}\, (\tau(\alpha) L^\alpha + \overline{\tau(\alpha) L^\alpha})^\perp \\ c^{-1} \frac{\lambda - \overline{\alpha}^{-1}}{\lambda - \alpha} & \text{on}\, \overline{\tau(\alpha) L^\alpha} \end{cases}$$

and

$$r_2(\lambda) := I \begin{cases} \frac{\lambda + \alpha}{\lambda + \overline{\alpha}^{-1}} & \text{on}\, \tau(\alpha) L^\alpha \\ 1 & \text{on}\, (\tau(\alpha) L^\alpha + \overline{\tau(\alpha) L^\alpha})^\perp \\ \frac{\lambda + \overline{\alpha}^{-1}}{\lambda + \alpha} & \text{on}\, \overline{\tau(\alpha) L^\alpha} \end{cases},$$

with $c := \frac{1 - \overline{\alpha}^{-2}}{1 - \alpha^2}$. After appropriate changes of variable and having in consideration (9.56) and (9.57), we conclude the holomorphicity of ξ at any of the other candidates for poles. $\square$

Thus:

Theorem 9.22. *When both are defined, twisted Bäcklund transformation of parameters α, L^α coincides with untwisted Bäcklund transformation of parameters α, L^α.*

Proposition 9.30 below provides a characterization, for the particular case α^2 is real, of untwisted Bäcklund transformations of parameters α, L^α defining twisted Bäcklund transformations of the same parameters.

9.3.3. Darboux transformation of constrained Willmore surfaces in 4-space. Characterized by the equation $d * (Q + q) = 0$, for some $q \in \Omega^1(\mathrm{End}_j(\underline{\mathbb{C}}^4/L, L))$ in certain conditions, a constrained Willmore surface L in S^4 ensures the existence of $G \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ with $dG = 2 * (Q + q)$, as well as the integrability of the Riccati

equation $dT = \rho T(dG)T - dF + 4\rho qT$, for each $\rho \in \mathbb{R}\backslash\{0\}$, fixing such a G and setting $F := G - S$. For a local solution $T \in \Gamma(Gl_j(\underline{\mathbb{C}}^4))$ of the ρ-Ricatti equation, we define the *constrained Willmore Darboux transform* of L of parameters ρ, T by setting $\hat{L} := T^{-1}L$, and extend, in this way, the Darboux transformation of Willmore surfaces in S^4 presented in [**12**] to a transformation of constrained Willmore surfaces in 4-space.

Consider $G \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$ with

$$dG = 2 * (Q + q)$$

(cf. Proposition 9.13) and set

$$F := G - S.$$

For $\rho \in \mathbb{R}\backslash\{0\}$, consider the ρ-Riccati equation

$$dT = \rho T(dG)T - dF + 4\rho qT. \tag{9.64}$$

Because L is q-constrained Willmore, we have

$$dq = [\mathcal{N}_S \wedge q] = \mathcal{N}_S \wedge q + q \wedge \mathcal{N}_S,$$

so that the integrability condition for equation (9.64),

$$\begin{aligned}
0 &= d(\rho T(dG)T - dF + 4\rho qT) \\
&= \rho\, dT \wedge (dG)T + \rho T(d^2G)T - \rho\, T(dG) \wedge dT - d^2F + 4\rho\, dq\, T - 4\rho\, q \wedge dT \\
&= \rho(\rho T(dG)T - dF + 4\rho\, qT) \wedge (dG)T - \rho\, T(dG) \wedge (\rho T(dG)T - dF + 4\rho\, qT) \\
&\quad + 4\rho\, dq\, T - 4\rho\, q \wedge (\rho T(dG)T - dF + 4\rho\, qT) \\
&= -\rho dF \wedge (dG)T + 4\rho^2 qT \wedge (dG)T + \rho T(dG) \wedge dF - 4\rho^2 T(dG) \wedge qT \\
&\quad + 4\rho\, [\mathcal{N}_S \wedge q]T - 4\rho^2 q \wedge T(dG)T + 4\rho\, q \wedge dF \\
&= -\rho dF \wedge (dG)T + \rho T(dG) \wedge dF - 4\rho^2 T(dG) \wedge qT + 4\rho\, [\mathcal{N}_S \wedge q]T + 4\rho\, q \wedge dF
\end{aligned}$$

is, equivalently, characterized by

$$\begin{aligned}
0 &= -4\rho\, (*A \wedge *Q + *A \wedge *q + *q \wedge *Q)T \\
&\quad + 4\rho\, T(*Q \wedge *A + *Q \wedge *q + *q \wedge *A) \\
&\quad - 8\rho^2 T * (Q + q) \wedge qT + 8\rho\, q \wedge *(A + q) \\
&\quad + 4\rho\, (A \wedge q + Q \wedge q + q \wedge A + q \wedge Q)T.
\end{aligned}$$

Obviously, given ω and γ 1-forms with values in a same bundle over M, $*\,\omega \wedge \gamma = -\omega \wedge *\,\gamma$ and, in particular, $*\,\omega \wedge *\,\gamma = \omega \wedge \gamma$. Hence equation (9.64) is integrable if and only if

$$\begin{aligned}
0 &= -4\rho\, *A \wedge *QT + 4\rho\, T(*Q \wedge *A + *Q \wedge *q + *q \wedge *A) \\
&\quad - 8\rho^2 T * (Q + q) \wedge qT + 8\rho\, q \wedge *(A + q) + 4\rho\, (Q \wedge q + q \wedge A)T.
\end{aligned}$$

We introduce now the concept of left and right-K and $\overline{K}$ type, which will prove very efficient in showing the vanishing of each of the terms above. Given $\xi \in \Omega^1(\mathrm{End}(\mathbb{C}^4))$, we say that ξ is of *left-K* (respectively, *right-K*) *type* if $*\,\xi = S\xi$ (respectively, $*\,\xi = \xi S$), referring to *left-$\overline{K}$* and *right-$\overline{K}$* type in the case $*\,\xi = -S\xi$ or, respectively, $*\,\xi = -\xi S$. For example, q is of both left-K and right-K type, whilst A is of left-K type and, therefore, given that it anti-commutes with S, of right-$\overline{K}$ type, as well; whereas Q is of both left-$\overline{K}$ and right-K type. It is obvious that the Hodge $*$-operator preserves types. Observe also that, given ξ_1 of right-K (respectively, right-$\overline{K}$) type and ξ_2 of left-K (respectively, left-$\overline{K}$) type, $\xi_1 \wedge \xi_2 = *\,\xi_1 \wedge *\,\xi_2 = \xi_1 S \wedge S\xi_2 = -\xi_1 \wedge \xi_2$, and, therefore, $\xi_1 \wedge \xi_2 = 0$. We conclude in this way the integrability of equation (9.64), ensuring the existence, for each $\rho \in \mathbb{R}\backslash\{0\}$, of a solution $T \in \Gamma(\mathrm{End}_j(\underline{\mathbb{C}}^4))$. Observe that the Riccati equation (9.64) has a conserved quantity, namely, given a solution T, if $(T-S)^2(p_0) = (\rho^{-1}-1)I$, at some $p_0 \in M$, then

$$(T-S)^2 = (\rho^{-1}-1)I \tag{9.65}$$

everywhere. In fact, setting

$$X_0 := (T-S)^2 - \rho^{-1}I + I = T^2 - TS - ST - \rho^{-1}I,$$

we have

$$\begin{aligned} dX_0 &= (dT-dS)(T-S) + (T-S)(dT-dS) \\ &= (\rho T(dG)T - dF + 4\rho qT - dS)T - (\rho T(dG)T - dF + 4\rho qT - dS)S \\ &\quad + T(\rho T(dG)T - dF + 4\rho qT - dS) - S(\rho T(dG)T - dF + 4\rho qT - dS) \end{aligned}$$

or, equivalently, as $dS = dG - dF$,

$$\begin{aligned} dX_0 &= (\rho T(dG)T - dG + 4\rho qT)T - (\rho T(dG)T - dG + 4\rho qT)S \\ &\quad + T(\rho T(dG)T - dG + 4\rho qT) - S(\rho T(dG)T - dG + 4\rho qT) \\ &= \rho\,T(dG)(T^2 - TS - \rho^{-1}I) + (T^2 - ST - \rho^{-1}I)\rho(dG)T \\ &\quad + 4\rho\,TqT + 4\rho\,q(T^2 - TS - ST - \rho^{-1}I), \end{aligned}$$

having in consideration that $S*Q = S(-SQ) = SQS = -(*\,Q)S$ and that, according to (9.25),

$$S*\,q = -q = (*\,q)\,S. \tag{9.66}$$

These relations make clear, on the other hand, that

$$\rho\,T(dG)ST + \rho\,TS(dG)T = \rho\,T(-4q)T.$$

We conclude that X_0 solves the first order linear equation

$$dX = \rho\,T(dG)X + X\rho(dG)T + 4\rho\,qX,$$

on X, and, therefore, that, if $X_0(p_0) = 0$ at some point $p_0 \in M$, then $X_0 = 0$. It follows that, by imposing, as initial condition,

$$T(p_0) = S(p_0) + I \begin{cases} \sqrt{\rho^{-1} - 1} & \text{on } W(p_0) \\ \overline{\sqrt{\rho^{-1} - 1}} & \text{on } jW(p_0) \end{cases},$$

for some $p_0 \in M$, some choice of $\sqrt{\rho^{-1} - 1}$, and some subspace $W(p_0)$ of $\mathbb{C}^4$, chosen as $\mathbb{C}^4$ itself, in the case $\rho \leq 1$, and as a non-j-stable 2-plane, otherwise; we define a local solution $T \in \Gamma(\mathrm{Gl}_j(\underline{\mathbb{C}}^4))$ of equation (9.64) with a conserved quantity satisfying equation (9.65). For such a solution T of the ρ-Riccati equation (9.64), we define the *constrained Willmore Darboux transform of* L *of parameters* ρ, T by setting

$$\hat{L} := T^{-1} L,$$

for T^{-1} the section of $\mathrm{End}_j(\underline{\mathbb{C}}^4)$ given by $T^{-1}(p) := T(p)^{-1}$, for all p. Observe that constrained Willmore Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial. In fact, if $\rho^{-1} - 1 \geq 0$, then $T = S + \sqrt{\rho^{-1} - 1}\, I$, for one of the square roots of $\rho^{-1} - 1$, and, therefore, $\hat{L} = L$, by equation (9.10). Of course, since L is a j-stable 2-plane, so is $\hat{L}$. As

$$dT^{-1} = -T^{-1}(dT)T^{-1} = -\rho dG + T^{-1}(dF)T^{-1} - 4\rho T^{-1} q$$

and both Q and q vanish on L, whilst

$$\mathrm{Im}\, dF \subset L,$$

we have

$$(dT^{-1})L \subset \hat{L}$$

and, therefore, given $l \in \Gamma(L)$,

$$d(T^{-1} l) + \hat{L} = (dT^{-1})l + T^{-1} dl + \hat{L} = T^{-1} dl + \hat{L},$$

showing that the derivative $\hat{\delta}$ of $\hat{L}$ relates to the one of L by

$$\hat{\delta} = T^{-1} \delta T_{|_{\hat{L}}}.$$

In particular, $\hat{L}$ is immersed. Equation (9.65) establishes, in particular,

$$0 = (T - S)^2 S - S(T - S)^2 = T^2 S - S T^2$$

and, therefore,

$$T^2 S = S T^2.$$

Thus

$$T^{-1} S T = T^{-1} S T^2 T^{-1} = T S T^{-1}.$$

Set then

$$\hat{S} := T S T^{-1} = T^{-1} S T,$$

which, as we verify next, consists of the mean curvature sphere of $\hat{L}$. Obviously, $\hat{S}\hat{L} = \hat{L}$ and $*\hat{\delta} = \hat{S} \circ \hat{\delta}$. According to (9.65), on the other hand,

$$T^2 = TS + ST + \rho^{-1} I, \tag{9.67}$$

so that

$$\hat{S} = T^{-1}(T^2 - TS - \rho^{-1}) = T - S - \rho^{-1}T^{-1} = F + T - (G + \rho^{-1}T^{-1})$$

and then

$$d\hat{S} = \rho(T(dG)T + 4qT) - (\rho^{-1}T^{-1}(dF)T^{-1} - 4T^{-1}q).$$

As $QL = 0 = qL$ and $\operatorname{Im} q$, $\operatorname{Im} A \subset L$, it is now clear that $(d\hat{S})\hat{L} \subset \hat{L}$, as well as, from $S * Q = Q$ and $S * A = -A$, that

$$\begin{aligned} \hat{S} d\hat{S} - * d\hat{S} &= \rho TS(dG)T + 4\rho TST^{-1}qT - \rho^{-1}T^{-1}S(dF)T^{-1} + 4T^{-1}Sq \\ &\quad - \rho T * (dG)T - 4\rho * qT + \rho^{-1}T^{-1}(*dF)T^{-1} - 4T^{-1} * q \\ &= 4\rho(TQT + TST^{-1}qT - *qT). \end{aligned}$$

Thus $(\hat{S}d\hat{S} - *d\hat{S})\hat{L} = 0$. This proves that $\hat{S}$ is the mean curvature sphere of $\hat{L}$, as well as that the Hopf fields of $\hat{L}$ relate to the Hopf fields of L by

$$\hat{Q} = \rho(TQT + TST^{-1}qT - *qT)$$

and, in view of the fact that $\hat{A} = \frac{1}{2} * d\hat{S} + \hat{Q}$,

$$\hat{A} = \rho TST^{-1}qT - \rho * qT - \rho TqT + 2\rho * qT + \rho^{-1}T^{-1}AT^{-1} + \rho^{-1}T^{-1}qT^{-1} + 2T^{-1} * q.$$

Aiming for a simpler relation between $\hat{A}$ and A, observe that, since $d\hat{S}$ is $\hat{S}$-anti-commuting,

$$\hat{S}d\hat{S} = \hat{S}(d \circ \hat{S} - \hat{S} \circ d) = -(d\hat{S})\hat{S},$$

$d\hat{S}$ reduces to the difference of the $\hat{S}$-anti-commuting parts of $\rho(T(dG)T + 4qT)$ and $\rho^{-1}T^{-1}(dF)T^{-1} - 4T^{-1}q$. The $\hat{S}$-anti-commuting part of $T(dG)T$ is

$$\frac{1}{2}(T(dG)T + \hat{S}T(dG)T\hat{S}) = T(*Q + S * QS + *q + S * qS)T = 2T * QT,$$

and, similarly, that of $T^{-1}(dF)T^{-1}$ is $2T^{-1} * AT^{-1}$; while the $\hat{S}$-anti-commuting parts of qT and $T^{-1}q$ are $\frac{1}{2}(qT + TST^{-1}qST)$ and $\frac{1}{2}(T^{-1}q + T^{-1}SqT^{-1}ST)$, respectively. It follows that

$$d\hat{S} = 2(*\hat{Q} - (\rho^{-1}T^{-1} * AT^{-1} - T^{-1}q - T^{-1} * qT^{-1}ST))$$

and, ultimately, that

$$\hat{A} = \rho^{-1}T^{-1}AT^{-1} - T^{-1} * q + T^{-1}qT^{-1}ST.$$

Set

$$\hat{q} := T^{-1}qT \in \Omega^1(\operatorname{End}(\underline{\mathbb{C}}^4/\hat{L}, \hat{L})).$$

Theorem 9.23. *$\hat{L}$ is a $\hat{q}$-constrained Willmore surface in S^4.*

Proof. Obviously, $\hat{S}\hat{q} = *\hat{q} = \hat{q}\hat{S}$. As we have previously observed, the $\hat{S}$-anti-commuting part of $d(F+T) = \rho(T(dG)T + 4qT)$ is $2*\hat{Q}$. On the other hand, as T^2 commutes with S, so does T^{-2},

$$ST^{-2} = T^{-2}T^2ST^{-2} = T^{-2}S,$$

which, together with equation (9.67), shows that

$$I = (TS + ST + \rho^{-1}I)T^{-2} = T^{-1}S + ST^{-1} + \rho^{-1}T^{-2}.$$

Thus

$$\begin{aligned}\hat{S}(T(dG)T + 4qT)\hat{S} &= TS(dG)ST + 4T(I - T^{-1}S - \rho^{-1}T^{-2})qST \\ &= TS(dG)ST + 4(TqST - SqST - \rho^{-1}T^{-1}qST).\end{aligned}$$

It follows that the $\hat{S}$-commuting part of $d(F+T)$ is

$$\begin{aligned}\frac{1}{2}(d(F+T) - \hat{S}d(F+T)\hat{S}) &= \rho T(*Q - S*QS + *q - S*qS)T \\ &\quad + 2\rho qT - 2\rho T*qT + 2\rho SqST + 2T^{-1}qST \\ &= 2T^{-1}*qT \\ &= 2*\hat{q}.\end{aligned}$$

Hence

$$d(F+T) = 2*(\hat{Q}+\hat{q}),$$

and, therefore,

$$d*(\hat{Q}+\hat{q}) = 0,$$

completing the proof. □

We complete this section by noting that

$$\hat{S}_+ = T^{-1}S_+ = TS_+, \quad \hat{S}_- = T^{-1}S_- = TS_-$$

and, consequently,

$$\hat{L}_+ = T^{-1}L_+, \quad \hat{L}_- = T^{-1}L_-.$$

9.3.4. Bäcklund transformation vs. Darboux transformation of constrained Willmore surfaces in 4-space. Constrained Willmore Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial. We establish a correspondence between constrained Willmore Darboux transformation parameters ρ, T with $\rho > 1$ and pairs $\alpha, L^\alpha; -\alpha, \rho_V L^\alpha$ of untwisted Bäcklund transformation parameters with α^2 real, and show that the corresponding transformations coincide. Non-trivial Darboux transformation of constrained Willmore surfaces in 4-space is, in this way, established as a

particular case of constrained Wilmore Bäcklund transformation.

Suppose $\rho > 1$ and T be constrained Willmore Darboux transformation parameters to L. Fix a choice of $\sqrt{\rho^{-1}-1}$. Then

$$T - S = I\begin{cases} \sqrt{\rho^{-1}-1} & \text{on}\, W \\ -\sqrt{\rho^{-1}-1} & \text{on}\, jW \end{cases},$$

for some non-j-stable bundle W of 2-planes in $\mathbb{C}^4$. Set

$$\alpha^2 := 2i\rho\sqrt{\rho^{-1}-1} - 2\rho + 1. \tag{9.68}$$

Lemma 9.24. *The bundle W is $d^S_{\alpha^2,q}$-parallel.*

Proof. For simplicity, set $\mu := \sqrt{\rho^{-1}-1}$ and $X := T - S$. Note that

$$\alpha^{-2} = -2i\rho\mu - 2\rho + 1.$$

Hence, according to Lemma 9.14,

$$\begin{aligned} d^S_{\alpha^2,q} &= d + 2\rho\mu * (Q+q) - 2\rho\,(Q+q) \\ &= d + \rho\mu\, dG + \rho * dG, \end{aligned}$$

as well as

$$d^S_{\alpha^{-2},q} = d - \rho\mu\, dG + \rho * dG.$$

On the one hand, straightforward computation establishes

$$\begin{aligned} &(\mu I - X)\circ d^S_{\alpha^2,q}\circ(X+\mu I) + (\mu I + X)\circ d^S_{\alpha^{-2},q}\circ(X-\mu I) = \\ &= 2\mu(dX + \rho(*\,dG)X + \rho\,\mu^2 dG - \rho X(dG)X - \rho X *\, dG), \end{aligned}$$

or, equivalently,

$$(\mu I - X)\circ\, d^S_{\alpha^2,q}\circ(X+\mu I) + (\mu I + X)\circ\, d^S_{\alpha^{-2},q}\circ(X-\mu I) = 2\mu(dX - \Phi), \tag{9.69}$$

for

$$\Phi := \rho\, X(dG)X + \rho[X, *dG] + (\rho - 1)dG.$$

On the other hand, in view of (9.66), together with $S * Q = Q = -(*\,Q)S$, we have

$$\begin{aligned} \Phi &= \rho\, T(dG)T - \rho\, T(dG)S - \rho\, S(dG)T + \rho\, S(dG)S \\ &\quad + \rho\, T * dG - \rho\, S * dG - \rho\,(*\,dG)T + \rho\,(*\,dG)S + \rho\, dG - dG \\ &= \rho\, T(dG)T + 2\rho\, TQ + 2\rho\, Tq - 2\rho\, QT + 2\rho\, qT - 2\rho\, SQ - 2\rho\, Sq - 2\rho\, TQ \\ &\quad - 2\rho\, Tq + 2\rho\, SQ + 2\rho\, Sq + 2\rho\, QT + 2\rho\, qT - 2\rho\, QS - 2\rho *\, q + 2\rho\, QS \\ &\quad + 2\rho *\, q - 2\, *\, Q - 2 *\, q \end{aligned}$$

and, therefore,

$$\Phi = \rho\, T(dG)T - dG + 4\rho\, qT, \tag{9.70}$$

or, equivalently, $dX = \Phi$. Thus

$$(\mu I - X) \circ d^S_{\alpha^2,q} \circ (X + \mu I) + (\mu I + X) \circ d^S_{\alpha^{-2},q} \circ (X - \mu I) = 0.$$

Obviously,

$$X + \mu I = 2\mu\pi_W, \quad X - \mu I = -2\mu\pi_{jW},$$

for $\pi_W : \underline{\mathbb{C}}^4 \to W$ and $\pi_{jW} : \underline{\mathbb{C}}^4 \to jW$ projections with respect to the decomposition $\underline{\mathbb{C}}^4 = W \oplus jW$. It follows that, given $\sigma \in \Gamma(W)$, $(\mu I - X) \circ d^S_{\alpha^2,q}\sigma = 0$, or, equivalently,

$$X d^S_{\alpha^2,q}\sigma = \mu\, d^S_{\alpha^2,q}\sigma,$$

completing the proof. □

For either choice of $\alpha = \sqrt{\alpha^2}$, define a null line subbundle of $\wedge^2\underline{\mathbb{C}}^4$ by

$$L^\alpha := \tau(\alpha)^{-1} \wedge^2 W. \tag{9.71}$$

Note that $L^{-\alpha} = \rho_V L^\alpha$. The $d^S_{\alpha^2,q}$-parallelness of W is equivalent to the $d^{\alpha,q}_V$-parallelness of L^α. The complementarity of W and jW in $\underline{\mathbb{C}}^4$, $\wedge^2 jW \cap (\wedge^2 W)^\perp = \{0\}$, is equivalent to the non-reality of $\tau(\alpha)L^\alpha$. Observe that $r(0) := r_{\sqrt{\alpha^2},\tau(\sqrt{\alpha^2})^{-1}\wedge^2 W}(0)$ and $T - S$ share eigenspaces:

$$r(0) = I \begin{cases} \sqrt{r_\alpha(0)} & \text{on } W \\ \sqrt{r_\alpha(0)}^{\,-1} & \text{on } jW \end{cases}.$$

Fixing a choice of $\sqrt{-\rho}$ according to the choices of $\sqrt{\rho^{-1} - 1}$ and $\sqrt{r_\alpha(0)}$, we get $\sqrt{r_\alpha(0)} = \sqrt{-\rho}\,(\sqrt{\rho^{-1} - 1} + i)$, and, consequently, $\sqrt{r_\alpha(0)}^{\,-1} = \sqrt{-\rho}\,(-\sqrt{\rho^{-1} - 1} + i)$. We conclude that

$$r(0) = \sqrt{-\rho}\,(T - S + i). \tag{9.72}$$

Hence

$$r(0)S_+ = \hat{S}_+,$$

establishing, in particular, the non-j-stability of $r(0)S_+$.

Note that, since $\sqrt{\rho^{-1} - 1}$ and $-\sqrt{\rho^{-1} - 1}$ are complex conjugate of each other, α^2 is real. Thus α^2 is unit if and only $\alpha^2 = \pm 1$, which, in view of $\rho \neq 1$, is impossible. It is immediate to verify that α^2 is non-zero.

Conversely, given a non-zero $\alpha \in \mathbb{C}\backslash S^1$, with α^2 real, and L^α a $d^{\alpha,q}_V$-parallel null line subbundle of $\wedge^2\underline{\mathbb{C}}^4$, with $\tau(\alpha)L^\alpha$ non-real, equation (9.68) determines

$$\rho = \frac{2\alpha^2 - 1 - \alpha^4}{4\alpha^2} > 1,$$

as well as a choice of $\sqrt{\rho^{-1}-1}$, whereas equation (9.71) determines a non-j-stable $d^S_{\alpha^2,q}$-parallel bundle W of 2-planes in $\mathbb{C}^4$. Obviously, the pair $(-\alpha, \rho_V L^\alpha)$ determines the same pair (ρ, W). Set

$$T := I\begin{cases} \sqrt{\rho^{-1}-1} & \text{on } W \\ -\sqrt{\rho^{-1}-1} & \text{on } jW \end{cases} + S.$$

Observe that

$$\overline{\tau(\alpha)\circ d_V^{\alpha,q}\circ\tau(\alpha)^{-1}\wedge^2 W} = \overline{\tau(\alpha)\circ d_V^{\alpha,q}\circ\tau(\alpha)^{-1}}\wedge^2 jW,$$

the $d^S_{\alpha^2,q}$-parallelness of W is equivalent to the parallelness of jW with respect to the connection

$$\overline{\tau(\alpha)\circ d_S^{\alpha,q}\circ\tau(\alpha)^{-1}} = \tau(\overline{\alpha}^{\,-1})\circ d_S^{\overline{\alpha}^{\,-1},q}\circ\tau(\overline{\alpha}) = d^S_{\overline{\alpha}^{\,-2},q}.$$

Given that α^2 is real, we conclude that jW is $d^S_{\alpha^{-2},q}$-parallel. It follows, in particular, that, for $\mu := \sqrt{\rho^{-1}-1}$ and $X := T - S$,

$$d^S_{\alpha^2,q}\circ(X+\mu I)\Gamma(\underline{\mathbb{C}}^4)\subset\Omega^1(W),\quad d^S_{\alpha^{-2},q}\circ(X-\mu I)\Gamma(\underline{\mathbb{C}}^4)\subset\Omega^1(jW),$$

or, equivalently,

$$(\mu I - X)\circ d^S_{\alpha^2,q}\circ(X+\mu I) = 0 = (\mu I + X)\circ d^S_{\alpha^{-2},q}\circ(X-\mu I).$$

From equations (9.69) and (9.70) - which derive solely from the fact that T satisfies equation (9.65), independently of T being a solution of equation (9.64) or not - we conclude that T is a solution of ρ-Riccati equation (9.64) (with a conserved quantity satisfying equation (9.65)).

This correspondence between constrained Willmore Darboux transformation parameters ρ, T with $\rho > 1$ and pairs α, L^α; $-\alpha, \rho_V L^\alpha$ of untwisted Bäcklund transformation parameters, with α^2 real, establishes, furthermore, a correspondence between transforms, as we verify next.

Suppose that the parameters α, L^α define an untwisted Bäcklund transform of L (i.e., that L^* immerses). Following (9.72), and in view of

$$\mathrm{Im}(-S+i)\subset S_-,$$

we conclude that, with respect to the decomposition of $\underline{\mathbb{C}}^4$ into the direct sum of S_+ and S_-,

$$\pi_{S_+} r(0)T^{-1}L_+ = L_+.$$

Equivalently,

$$\pi_{S_+^*} r(\infty)L_+ = T^{-1}L_+,$$

with respect to the decomposition of $\underline{\mathbb{C}}^4$ into the direct sum of S_+^* and S_-^*. In fact, given $l_+ \in \Gamma(L_+)$, $\pi_{S_+^*} r(\infty)l_+$ is a section of $T^{-1}L_+$ if and only if, for some $\lambda \in \mathbb{C}$,

$r(\infty)l_+ - \lambda T^{-1}l_+$ is a section of $r(\infty)S_-$, or, equivalently, recalling (9.40),

$$l_+ = \lambda \pi_{S_+}(r(0)T^{-1}l_+).$$

We conclude that $L^*_+ = \hat{L}_+$, or, equivalently,

$$L^* = \hat{L}.$$

If L^* does not immerse, we define the untwisted Bäcklund transform of L of parameters α, L^α to be $\hat{L}$. In this sense, we have just proven the following:

Theorem 9.25. *Constrained Willmore Darboux transformation of parameters ρ, T with $\rho > 1$ is equivalent to untwisted Bäcklund transformation of parameters α, L^α with α^2 real. Constrained Willmore Darboux transformation of parameters ρ, T with $\rho \leq 1$ is trivial.*

In particular, twisted Bäcklund transforms of parameters α, L^α with α^2 real, $\tau(\alpha)L^\alpha$ non-real and $r(0)S_+$ non-j-stable, are constrained Willmore Darboux transforms. Next we examine what they correspond to under the correspondence established above between untwisted Bäcklund transformation of parameters α, L^α with α^2 real and constrained Willmore Darboux transformation of parameters ρ, T with $\rho > 1$.

In what follows, let ρ, T be constrained Willmore Darboux transformation parameters, with $\rho > 1$, and α, L^α be corresponding untwisted Bäcklund transformation parameters, under the correspondence established above.

Lemma 9.26. *L^α is orthogonal to $\rho_V L^\alpha$ at a point in M if and only if, at that point, either $W \cap S_+ \neq \{0\}$ or $W \cap S_- \neq \{0\}$.*

Before proceeding to the proof of the lemma, it is opportune to emphasize the following fact.

Lemma 9.27. *Let $w_1 := w_1^+ + w_1^-, w_2 := w_2^+ + w_2^-$ be a frame of W with $w_i^\pm \in \Gamma(S_\pm)$, respectively, for $i = 1, 2$. Then, at each point, $W \cap S_\pm \neq \{0\}$ if and only if $w_1^\mp \wedge w_2^\mp = 0$, respectively.*

Proof. Let p be a point in M. Let $w := aw_1 + bw_2 \in \Gamma(W)$, with $a, b \in \Gamma(\underline{\mathbb{C}})$, be not-zero at p. Then $w(p)$ is in $S_\pm(p)$ if and only if, at the point p, one has either $b \neq 0$ and $w_2^\mp = -\frac{a}{b} w_1^\mp$ or $b = 0 \neq a$ and $w_1^\mp = 0$, respectively. On the other hand, if, at p, $w_1^\mp \wedge w_2^\mp = 0$, then either $w_i(p) \in W(p) \cap S_\pm(p)$, respectively, for some $i = 1, 2$; or $w_2^\mp(p) = \lambda w_1^\mp(p)$, for some $\lambda \in \mathbb{C}\backslash\{0\}$, in which case $-\lambda w_1(p) + w_2(p) \in W(p) \cap S_\pm(p)\backslash\{0\}$, respectively.

$\square$

Now we proceed to the proof of Lemma 9.26.

Proof. Let $w_1 := w_1^+ + w_1^-, w_2 := w_2^+ + w_2^-$ be a frame of W with $w_i^\pm \in \Gamma(S_\pm)$, respectively, for $i = 1, 2$. Recall that $\rho_V = \tau(-1)$. At each point, the orthogonality of L^α to $\rho_V L^\alpha$ is characterized by

$$(\rho(w_1 \wedge w_2), w_1 \wedge w_2) = 0,$$

or, equivalently,

$$(w_1^+ \wedge w_2^+, w_1^- \wedge w_2^-) = 0. \tag{9.73}$$

On the other hand, at each point, equation (9.73) holds if and only if either $w_1^+ \wedge w_2^+ = 0$ or $w_1^- \wedge w_2^- = 0$, which, according to Lemma 9.27, completes the proof. $\square$

Lemma 9.28. *If, at some point , $W \cap S_+ = \{0\} = W \cap S_-$, then, at that point, $q_{\overline{\alpha}^{-1}, \overline{L^\alpha}}(\alpha) L^\alpha$ is not orthogonal to $\rho_V q_{\overline{\alpha}^{-1}, \overline{L^\alpha}}(\alpha) L^\alpha$.*

Proof. Our argument is pointwise, so we work in a single fibre.

Suppose $W \cap S_+ = \{0\} = W \cap S_-$. In that case, according to Lemma 9.26, $\overline{L^\alpha}$ is not orthogonal to $\rho_V \overline{L^\alpha}$, or, equivalently, $(\wedge^2 \tau(\overline{\alpha}) jW) \cap (\wedge^2 \tau(-\overline{\alpha}) jW)^\perp = \{0\}$, which characterizes the complementarity of $\tau(\overline{\alpha}) jW$ and $\tau(-\overline{\alpha}) jW$ in $\underline{\mathbb{C}}^4$. Set $A := \frac{\alpha - \overline{\alpha}^{-1}}{\alpha + \overline{\alpha}^{-1}}$ and fix a choice of $\sqrt{A}$. Then

$$q := q_{\overline{\alpha}^{-1}, \overline{L^\alpha}}(\alpha) = I \begin{cases} \sqrt{A} & \text{on } \tau(\overline{\alpha}) jW \\ \sqrt{A}^{\,-1} & \text{on } \tau(-\overline{\alpha}) jW \end{cases}.$$

Let $w_1 := w_1^+ + w_1^-, w_2 := w_2^+ + w_2^-$ be a frame of W with $w_i^\pm \in \Gamma(S_\pm)$, respectively, for $i = 1, 2$. L^α is spanned by

$$\begin{aligned} l^\alpha &:= \tau(\alpha)^{-1}(w_1 \wedge w_2) \\ &= \alpha\, w_1^+ \wedge w_2^+ + w_1^+ \wedge w_2^- + w_1^- \wedge w_2^+ + \alpha^{-1} w_1^- \wedge w_2^-. \end{aligned}$$

The proof will consist of showing that $(\rho_V q l^\alpha, q l^\alpha) \neq 0$, or, equivalently, in view of the nullity of $q l^\alpha$, that

$$(\pi_{\wedge^2 S_+} q l^\alpha, \pi_{\wedge^2 S_-} q l^\alpha) \neq 0,$$

for $\pi_{\wedge^2 S_+} : \wedge^2 \underline{\mathbb{C}}^4 \to \wedge^2 S_+$ and $\pi_{\wedge^2 S_-} : \wedge^2 \underline{\mathbb{C}}^4 \to \wedge^2 S_-$ projections with respect to decomposition (9.15).

Fix a choice of $\sqrt{\alpha}$. Then

$$\tau(\overline{\alpha}) = I \begin{cases} \overline{\sqrt{\alpha}}^{\,-1} & \text{on } S_+ \\ \overline{\sqrt{\alpha}} & \text{on } S_- \end{cases},$$

whereas

$$\tau(-\overline{\alpha}) = I \begin{cases} i\overline{\sqrt{\alpha}}^{\,-1} & \text{on } S_+ \\ -i\overline{\sqrt{\alpha}} & \text{on } S_- \end{cases},$$

and, therefore,

$$\tau(\overline{\alpha})jW = \langle u_1 := \overline{\alpha}jw_1^+ + jw_1^-, u_2 := \overline{\alpha}jw_2^+ + jw_2^-\rangle,$$

whereas

$$\tau(-\overline{\alpha})jW = \langle v_1 := -\overline{\alpha}jw_1^+ + jw_1^-, v_2 := -\overline{\alpha}jw_2^+ + jw_2^-\rangle.$$

Given $i = 1, 2$ and $a_i^\pm, b_i^\pm, c_i^\pm, d_i^\pm \in \Gamma(\underline{\mathbb{C}})$ for which

$$w_i^\pm = a_i^\pm u_1 + b_i^\pm u_2 + c_i^\pm v_1 + d_i^\pm v_2,$$

respectively, the fact that $W \cap S_+ = \{0\} = W \cap S_-$, or, equivalently,

$$jw_1^+ \wedge jw_2^+ \neq 0 \neq jw_1^- \wedge jw_2^-, \tag{9.74}$$

forces

$$c_i^+ = a_i^+,\ d_i^+ = b_i^+,\ c_i^- = -a_i^-,\ d_i^- = -b_i^-. \tag{9.75}$$

Hence

$$\begin{aligned} qw_i^\pm &= a_i^\pm\overline{\alpha}(\sqrt{A} \mp \sqrt{A}^{-1})jw_1^+ + a_i^\pm(\sqrt{A} \pm \sqrt{A}^{-1})jw_1^- \\ &\quad + b_i^\pm\overline{\alpha}(\sqrt{A} \mp \sqrt{A}^{-1})jw_2^+ + b_i^\pm(\sqrt{A} \pm \sqrt{A}^{-1})jw_2^-, \end{aligned}$$

respectively, for $i = 1, 2$; and, therefore,

$$\begin{aligned} qw_1^\pm \wedge qw_2^\pm &= (a_1^\pm b_2^\pm - b_1^\pm a_2^\pm)\,\alpha^2(\sqrt{A} \mp \sqrt{A}^{-1})^2 jw_1^+ \wedge jw_2^+ \\ &\quad + (a_1^\pm b_2^\pm - b_1^\pm a_2^\pm)\,\overline{\alpha}\,(A - A^{-1})(jw_1^+ \wedge jw_2^- + jw_1^- \wedge jw_2^+) \\ &\quad + (a_1^\pm b_2^\pm - b_1^\pm a_2^\pm)\,(\sqrt{A} \pm \sqrt{A}^{-1})^2 jw_1^- \wedge jw_2^-, \end{aligned}$$

respectively, as well as

$$\begin{aligned} qw_i^+ \wedge qw_k^- &= (A - A^{-1})(a_i^+ b_k^- - b_i^+ a_k^-)(\alpha^2 jw_1^+ \wedge jw_2^+ + jw_1^- \wedge jw_2^-) \\ &\quad + \overline{\alpha}\, a_i^+ a_k^-((\sqrt{A} - \sqrt{A}^{-1})^2 - (\sqrt{A} + \sqrt{A}^{-1})^2) jw_1^+ \wedge jw_1^- \\ &\quad + \overline{\alpha}\,(a_i^+ b_k^-(\sqrt{A} - \sqrt{A}^{-1})^2 - b_i^+ a_k^-(\sqrt{A} + \sqrt{A}^{-1})^2) jw_1^+ \wedge jw_2^- \\ &\quad + \overline{\alpha}\,(a_i^+ b_k^-(\sqrt{A} + \sqrt{A}^{-1})^2 - b_i^+ a_k^-(\sqrt{A} - \sqrt{A}^{-1})^2) jw_1^- \wedge jw_2^+ \\ &\quad + \overline{\alpha}\, b_i^+ b_k^-((\sqrt{A} - \sqrt{A}^{-1})^2 - (\sqrt{A} + \sqrt{A}^{-1})^2) jw_2^+ \wedge jw_2^-, \end{aligned}$$

for $i \neq k$. It follows that

$$\pi_{\wedge^2 S_+} ql^\alpha = X_+\, jw_1^- \wedge jw_2^-$$

and

$$\pi_{\wedge^2 S_-} ql^\alpha = X_-\, jw_1^+ \wedge jw_2^+,$$

for

$$\begin{aligned} X_+ &= \alpha(\sqrt{A} + \sqrt{A}^{-1})^2(a_1^+ b_2^+ - b_1^+ a_2^+) + \alpha^{-1}(\sqrt{A} - \sqrt{A}^{-1})^2(a_1^- b_2^- - b_1^- a_2^-) \\ &\quad + (A - A^{-1})(a_1^+ b_2^- - b_1^+ a_2^- + a_1^- b_2^+ - b_1^- a_2^+) \end{aligned}$$

and

$$\begin{aligned}\alpha^{-2}X_- &= \alpha(\sqrt{A}-\sqrt{A}^{\,-1})^2(a_1^+b_2^+ - b_1^+a_2^+) + \alpha^{-1}(\sqrt{A}+\sqrt{A}^{\,-1})^2(a_1^-b_2^- - b_1^-a_2^-)\\ &\quad + (A-A^{-1})(a_1^+b_2^- - b_1^+a_2^- + a_1^-b_2^+ - b_1^-a_2^+).\end{aligned}$$

According to (9.74),

$$(jw_1^+\wedge jw_2^+, jw_1^-\wedge jw_2^-)\neq 0,$$

so that $(\rho_V ql^\alpha, ql^\alpha)$ vanishes if and only if $X_+\,\alpha^{-2}X_-$ does. As α^2 is real, or, equivalently, α is either real or pure imaginary, we have

$$\alpha^{\pm 1}(\sqrt{A}\pm\sqrt{A}^{\,-1})^2 = \frac{4\alpha^{\pm 3}}{\alpha^2-\alpha^{-2}}$$

and

$$\alpha^{\pm 1}(\sqrt{A}\mp\sqrt{A}^{\,-1})^2 = \frac{4\alpha^{\mp 1}}{\alpha^2-\alpha^{-2}},$$

respectively, whilst, depending, respectively, on α being real or pure imaginary,

$$A - A^{-1} = \mp\frac{4}{\alpha^2-\alpha^{-2}}.$$

Hence the orthogonality of $\rho_V ql^\alpha$ to ql^α is characterized, equivalently, by the equation

$$\begin{aligned}0 &= \alpha^2(a_1^+b_2^+ - b_1^+a_2^+)^2 + (\alpha^4+\alpha^{-4})(a_1^+b_2^+ - b_1^+a_2^+)(a_1^-b_2^- - b_1^-a_2^-)\\ &\quad \mp(\alpha^3+\alpha^{-1})(a_1^+b_2^+ - b_1^+a_2^+)(a_1^+b_2^- - b_1^+a_2^- + a_1^-b_2^+ - b_1^-a_2^+)\\ &\quad \mp(\alpha^{-3}+\alpha)(a_1^-b_2^- - b_1^-a_2^-)(a_1^+b_2^- - b_1^+a_2^- + a_1^-b_2^+ - b_1^-a_2^+)\\ &\quad + \alpha^{-2}(a_1^-b_2^- - b_1^-a_2^-)^2 + (a_1^+b_2^- - b_1^+a_2^- + a_1^-b_2^+ - b_1^-a_2^+)^2,\end{aligned}$$

depending on α being real or pure imaginary, respectively. Observe now that, on the other hand, according to (9.75),

$$w_i^+ = 2\,(a_i^+ jw_1^- + b_i^+ jw_2^-),\quad w_i^- = 2\,\overline{\alpha}\,(a_i^- jm_1^+ + b_i^- jw_2^+),$$

and, therefore,

$$jw_i^+ = -2\,(\overline{a_i^+}\,w_1^- + \overline{b_i^+}\,w_2^-),\quad jw_i^- = -2\,\alpha\,(\overline{a_i^-}\,w_1^+ + \overline{b_i^-}\,w_2^+),$$

for $i = 1, 2$. It follows, in particular, that

$$jw_1^+\wedge jw_2^+ = 4\,\overline{(a_1^+b_2^+ - b_1^+a_2^+)}\,w_1^-\wedge w_2^-$$

and, consequently, that

$$(jw_1^+\wedge jw_2^+, w_1^+\wedge w_2^+) = 4\,\overline{(a_1^+b_2^+ - b_1^+a_2^+)}\,(w_1^-\wedge w_2^-, w_1^+\wedge w_2^+),$$

or, equivalently,

$$a_1^+b_2^+ - b_1^+a_2^+ = \frac{1}{4}\frac{(w_1^+\wedge w_2^+, jw_1^+\wedge jw_2^+)}{(jw_1^-\wedge jw_2^-, jw_1^+\wedge jw_2^+)}.$$

Similarly, from

$$jw_1^- \wedge jw_2^- = 4\,\alpha^2\overline{(a_1^- b_2^- - b_1^- a_2^-)}\, w_1^+ \wedge w_2^+,$$

we conclude that

$$a_1^- b_2^- - b_1^- a_2^- = \frac{1}{4\alpha^2}\frac{(w_1^- \wedge w_2^-, jw_1^- \wedge jw_2^-)}{(jw_1^+ \wedge jw_2^+, jw_1^- \wedge jw_2^-)}.$$

On the other hand,

$$jw_1^+ \wedge jw_2^- = 4\,\alpha\,(\overline{a_1^+ a_2^-}\, w_1^- \wedge w_1^+ + \overline{a_1^+ b_2^-}\, w_1^- \wedge w_2^+ + \overline{b_1^+ a_2^-}\, w_2^- \wedge w_1^+ + \overline{b_1^+ b_2^-}\, w_2^- \wedge w_2^+),$$

showing that

$$a_1^+ b_2^- = -\frac{1}{4\overline{\alpha}}\frac{(w_1^+ \wedge w_2^-, jw_1^+ \wedge jw_2^-)}{(jw_1^+ \wedge jw_2^+, jw_1^- \wedge jw_2^-)},$$

as well as

$$b_1^+ a_2^- = \frac{1}{4\overline{\alpha}}\frac{(w_1^+ \wedge w_2^-, jw_1^- \wedge jw_2^+)}{(jw_1^+ \wedge jw_2^+, jw_1^- \wedge jw_2^-)}.$$

Similarly, the fact that

$$jw_1^- \wedge jw_2^+ = 4\,\alpha\,(\overline{a_1^- a_2^+} w_1^+ \wedge w_1^- + \overline{a_1^- b_2^+} w_1^+ \wedge w_2^- + \overline{b_1^- a_2^+} w_2^+ \wedge w_1^- + \overline{b_1^- b_2^+} w_2^+ \wedge w_2^-)$$

establishes

$$a_1^- b_2^+ = -\frac{1}{4\overline{\alpha}}\frac{(w_1^- \wedge w_2^+, jw_1^- \wedge jw_2^+)}{(jw_1^+ \wedge jw_2^+, jw_1^- \wedge jw_2^-)}$$

and

$$b_1^- a_2^+ = \frac{1}{4\overline{\alpha}}\frac{(w_1^- \wedge w_2^+, jw_1^+ \wedge jw_2^-)}{(jw_1^+ \wedge jw_2^+, jw_1^- \wedge jw_2^-)}.$$

Set

$$Y_\pm := (w_1^\pm \wedge w_2^\pm, jw_1^\pm \wedge jw_2^\pm),$$

respectively, and

$$\begin{aligned} Y \;&:=\; (w_1^+ \wedge w_2^-, jw_1^+ \wedge jw_2^-) + (w_1^+ \wedge w_2^-, jw_1^- \wedge jw_2^+) \\ &\quad + (w_1^- \wedge w_2^+, jw_1^- \wedge jw_2^+) + (w_1^- \wedge w_2^+, jw_1^+ \wedge jw_2^-). \end{aligned}$$

It follows that $\rho_V l^\alpha$ is orthogonal to ql^α if and only if

$$\alpha^2 Y_+^2 + (\alpha^2 + \alpha^{-6}) Y_+ Y_- + \alpha^{-6} Y_-^2 \pm \frac{\alpha^3 + \alpha^{-1}}{\overline{\alpha}} Y_+ Y \pm \frac{\alpha^{-3} + \alpha}{\alpha^2 \overline{\alpha}} Y_- Y + \alpha^{-2} Y^2 = 0,$$

depending on α being real or pure imaginary, respectively. But, obviously, depending on α being real or pure imaginary, respectively,

$$\frac{\alpha^3 + \alpha^{-1}}{\overline{\alpha}} = \pm\,(\alpha^2 + \alpha^{-2}), \quad \frac{\alpha^{-3} + \alpha}{\alpha^2 \overline{\alpha}} = \pm\,(\alpha^{-6} + \alpha^{-2}).$$

Set

$$\hat{Y} := Y + Y_+ + Y_- = (w_1 \wedge w_2, jw_1 \wedge jw_2) \neq 0.$$

To complete the proof, we are left to verify that

$$\alpha^2 Y_+^2 + \alpha^{-6} Y_-^2 + \alpha^{-2} Y^2 + \alpha^2 Y_+(\hat{Y} - Y_+) + \alpha^{-2} Y(\hat{Y} - Y) + \alpha^{-6} Y_-(\hat{Y} - Y_-) \neq 0,$$

or, equivalently, as $\hat{Y}$ is not zero, that

$$\alpha^2 Y_+ + \alpha^{-2} Y + \alpha^{-6} Y_- \neq 0.$$

According to (9.7), for $i \neq k$,

$$(w_i^+ \wedge w_k^-, jw_i^+ \wedge jw_k^-) \geq 0,$$

and, therefore,

$$Y \geq 0.$$

On the other hand, the fact that $W \cap S_+ = \{0\} = W \cap S_-$, or, equivalently,

$$w_1^+ \wedge w_2^+ \neq 0 \neq w_1^- \wedge w_2^-,$$

intervenes, yet again, to ensure, according to Remark 9.3, that

$$Y_\pm > 0.$$

The fact that $\alpha^2 \in \mathbb{R}\backslash\{0\}$ completes the proof. □

Lemma 9.29. *The following are equivalent, respectively and pointwise:*

i) $W \cap S_\pm \neq \{0\}$;

ii) $L^\alpha \subset \wedge^2 S_\pm \oplus S_\pm \wedge S_\mp$;

iii) $\wedge^2 S_\pm \cap (L^\alpha)^\perp \neq \{0\}$.

Proof. Since W has rank 2, or, equivalently, $\wedge^2 W$ has rank 1, it is clear that $W \cap S_\pm \neq \{0\}$ if and only if $\wedge^2 W \subset \wedge^2 S_\pm \oplus S_\pm \wedge S_\mp$, respectively. The fact that $\tau(\alpha)$ preserves $\wedge^2 S_\pm$ and $S_\pm \wedge S_\mp$ establishes then the equivalence of *i*) and *ii*). The equivalence of *ii*) and *iii*) is obvious, in view of the complementarity of S_+ and S_- in $\mathbb{C}^4$. Lastly, it is obvious that a subbundle of $\wedge^2 \underline{\mathbb{C}}^4$ is orthogonal to $\wedge^2 S_\pm$ if and only if its projection onto $\wedge^2 S_\mp$, respectively, with respect to the decomposition (9.15), is zero. The fact that L^α is a line bundle completes the proof. □

We refer to an untwisted Bäcklund transformation of parameters α, L^α for which, locally,

$$L^\alpha \nsubseteq \wedge^2 S_+ \oplus S_+ \wedge S_- \quad \text{and} \quad L^\alpha \nsubseteq \wedge^2 S_- \oplus S_+ \wedge S_-,$$

as *regular*; as well as to a constrained Willmore Darboux transformation of parameters ρ, T for which, locally,

$$W \cap S_+ = \{0\} = W \cap S_-,$$

for W the eigenspace of $T - S$ associated to the eigenvalue $\sqrt{\rho^{-1} - 1}$. The regularity of an untwisted Bäcklund transformation of parameters α, L^α with α^2 real is equivalent to the regularity of the corresponding Darboux transformation of parameters ρ, T with

$\rho > 1$. The fact that regularity can, equivalently, be characterized by

$$\wedge^2 S_+ \cap (L^\alpha)^\perp = \{0\} = \wedge^2 S_- \cap (L^\alpha)^\perp$$

makes clear that it is an open condition on the points of M.

According to Lemmas 9.26 and 9.28, we have:

Proposition 9.30. *An untwisted Bäcklund transformation of parameters α, L^α with α^2 real defines a twisted Bäcklund transformation of parameters α, L^α if and only if it is regular.*

Hence, following Theorem 9.25:

Proposition 9.31. *Twisted Bäcklund transformation of parameters α, L^α with α^2 real, $\tau(\alpha)L^\alpha$ non-real and $r(0)S_+$ non-j-stable is equivalent to regular constrained Willmore Darboux transformation of parameters ρ, T with $\rho > 1$.*

APPENDIX A

Hopf differential and umbilics

According to (5.2), given z and ω holomorphic charts of $(M, \mathcal{C}_\Lambda)$, k^z vanishes if and only if k^ω does.

Proposition A.1. *Suppose $\Lambda \subset \underline{\mathbb{R}}^{4,1}$ is an isothermic surface in 3-space. Then, given $v_\infty \in \mathbb{R}^{4,1}$ light-like, the umbilic points of the surface in S_{v_∞} defined by Λ are the points at which the Hopf differential of Λ vanishes.*

To prove the proposition, we start by establishing a relation between the Hopf differential of Λ and the (classical) Hopf differential of σ^z:

Lemma A.2. *Let $v_\infty \in \mathbb{R}^{n+1,1}$ be non-zero, σ_∞ be the surface defined by Λ in S_{v_∞}, and ξ be a normal vector field to σ_∞. Let z be a holomorphic chart of M. Then*

$$(\sigma^z_{zz}, \xi) = \lambda(k^z, \mathcal{Q}\xi),$$

for $\mathcal{Q} : N_\infty \to S^\perp$ the isometry defined in Section 2.2 and $\lambda \in \Gamma(\underline{\mathbb{R}})$ defined by $\sigma^z = \lambda\sigma_\infty$.

Proof. Recall, yet again, that $\sigma^*_\infty TS_{v_\infty}$ consists of the orthogonal complement in $\underline{\mathbb{R}}^{n+1,1}$ of the non-degenerate bundle $\langle \sigma_\infty, v_\infty \rangle$. Let π_{N_∞} denote the orthogonal projection of $\underline{\mathbb{R}}^{n+1,1} = d\sigma_\infty(TM) \oplus N_\infty \oplus \langle v_\infty, \sigma_\infty \rangle$ onto N_∞. Following equation (8.5), we have $((\sigma_\infty)_{zz}, \xi) = (\pi_{N_\infty}(\sigma_\infty)_{zz}, \xi) = (\pi_{S^\perp}(\sigma_\infty)_{zz}, \mathcal{Q}\xi)$. On the other hand, writing $\sigma^z = \lambda\sigma_\infty$ with $\lambda \in \Gamma(\underline{\mathbb{R}})$, we have $\sigma^z_{zz} = \lambda(\sigma_\infty)_{zz} + 2\lambda_z(\sigma_\infty)_z + \lambda_{zz}\sigma_\infty$ and, therefore,

$$(\sigma^z_{zz}, \xi) = \lambda((\sigma_\infty)_{zz}, \xi), \tag{A.1}$$

which, together with equation (8.6), completes the proof. □

Now suppose $\Lambda \subset \underline{\mathbb{R}}^{4,1}$ is an isothermic surface in 3-space. Let $v_\infty \in \mathbb{R}^{4,1}$ be light-like and σ_∞ be the surface in S_{v_∞} defined by Λ. Cf. Theorem 8.6, σ_∞ is isothermic. Let x, y be conformal curvature line coordinates of σ_∞. Fix $\xi \in \Gamma(N_\infty)$ unit. Let A^ξ_∞ denote the shape operator of σ_∞ with respect to ξ. Then

$$A^\xi_\infty(\delta_x) = k_1\delta_x, \quad A^\xi_\infty(\delta_y) = k_2\delta_y,$$

for some $k_1, k_2 \in \Gamma(\underline{\mathbb{R}})$, the principal curvatures of σ_∞, locally, in the domain of the chart $z := x + iy$ of M. In view of the conformality of the coordinates x and y, z is, up to a change of orientation in M, a holomorphic chart of $(M, \mathcal{C}_\Lambda)$. In these conditions:

Lemma A.3. *The (local) principal curvatures, k_1 and k_2, of σ_∞ relate to the Hopf differential of Λ by*

$$e^u(k_1 - k_2) = 4(k^z, \mathcal{Q}\xi),$$

with $u \in \Gamma(\underline{\mathbb{R}})$.

Proof. The fact that x, y are conformal, $dx^2 + dy^2 \in \mathcal{C}_\Lambda \ni g_\infty$, establishes, in particular,

$$g_\infty(\delta_x, \delta_x) = e^u = g_\infty(\delta_y, \delta_y),$$

for some $u \in \Gamma(\underline{\mathbb{R}})$, and, therefore,

$$g_\infty(A^\xi_\infty(\delta_x), \delta_x) - g_\infty(A^\xi_\infty(\delta_y), \delta_y) = e^u(k_1 - k_2).$$

On the other hand, according to equation (2.2),

$$g_\infty(A^\xi_\infty(\delta_x), \delta_x) - g_\infty(A^\xi_\infty(\delta_y), \delta_y) = (\Pi_\infty(\delta_x, \delta_x), \xi) - (\Pi_\infty(\delta_y, \delta_y), \xi).$$

Since the conformal coordinates x, y are curvature line, it follows, by equation (8.1), that

$$e^u(k_1 - k_2) = 4(\Pi_\infty(\delta_z, \delta_z), \xi) = 4(\pi_{N_\infty}(\sigma_\infty)_{zz}, \xi),$$

for π_{N_∞} the orthogonal projection of $\underline{\mathbb{R}}^{4,1} = d\sigma_\infty(TM) \oplus N_\infty \oplus \langle v_\infty, \sigma_\infty \rangle$ onto

$$N_\infty \subset \sigma^*_\infty TS_{v_\infty} = \langle v_\infty, \sigma_\infty \rangle^\perp,$$

and, consequently, $e^u(k_1 - k_2) = 4((\sigma_\infty)_{zz}, \xi)$. Now write $\sigma^z = \lambda\,\sigma_\infty$, with $\lambda \in \Gamma(\underline{\mathbb{R}})$. By equation (A.1), we get $e^u(k_1 - k_2) = 4\lambda^{-1}(\sigma^z_{zz}, \xi)$ and the conclusion follows then from Lemma A.2. □

Proposition A.1 follows.

APPENDIX B

Twisted vs. untwisted Bäcklund transformation parameters

Twisted and untwisted Bäcklund transformation parameters conditions at a point are not equivalent. On the one hand, the choice of L^α as a null line bundle defined naturally by $d_V^{\alpha,q}$-parallel transport of l_p^α, for l^α a non-zero section of $\wedge^2 S_+$ and p a point in M, establishes the existence of untwisted Bäcklund transformation parameters α, L^α, at the point p, with $\rho_V L^\alpha$ orthogonal to L^α at p. On the other hand, the choice of L^α as a null line bundle defined naturally by $d_V^{\alpha,q}$-parallel transport of l_p^α, for some point $p \in M$ and some $l^\alpha := v_0 + v_+ + v_- \in \Gamma(\wedge^2 \underline{\mathbb{C}}^4)$ with $v_0 \in \Gamma(S_+ \wedge S_-)$ purely imaginary unit, $v_- \in \Gamma(\wedge^2 S_-)$ with

$$(v_-, \overline{v_-}) = \frac{1}{2} \mid \alpha \mid^{-2}, \tag{B.1}$$

and $v_+ = -\frac{1}{2}(v_-, \overline{v_-})^{-1}\overline{v_-}$, establishes the existence of twisted Bäcklund transformation parameters α, L^α, at the point p, with $\tau(\alpha)L^\alpha$ real at p. Indeed, the conditions on v_0, v_- and v_+ ensure immediately the non-orthogonality of L^α and $\rho_V L^\alpha$ and the reality of $\tau(\alpha)L^\alpha$, at the point p, which, together, ensure the non-orthogonality of $\tilde{L}^\alpha$ and $\rho_V \tilde{L}^\alpha$ at p, as we verify next. We work in the fibre at p. Computation shows that, as L^α is null and not orthogonal to $\rho_V L^\alpha$,[1]

$$q(\alpha)l^\alpha = v_0 + X_0\overline{v_0} + X_- v_- + X_+\overline{v_-}$$

for

$$\begin{aligned} X_0 &= A\Big(\frac{(v_0,\overline{v_0})}{2(v_0,v_0)} - \frac{(v_-,\overline{v_-})}{2(v_0,v_0)} - \frac{(v_0,v_0)}{8(v_-,\overline{v_-})}\Big) \\ &\quad - \frac{(v_0,\overline{v_0})}{(v_0,v_0)} + A^{-1}\Big(\frac{(v_0,\overline{v_0})}{2(v_0,v_0)} + \frac{(v_-,\overline{v_-})}{2(v_0,v_0)} + \frac{(v_0,v_0)}{8(v_-,\overline{v_-})}\Big), \\ X_- &= A\Big(\frac{\mid (v_0,v_0)\mid^2}{16(v_-,\overline{v_-})^2} - \frac{(v_0,\overline{v_0})}{4(v_-,\overline{v_-})} + \frac{1}{4}\Big) + \frac{1}{2} \\ &\quad - \frac{\mid (v_0,v_0)\mid^2}{8(v_-,\overline{v_-})^2} + A^{-1}\Big(\frac{\mid (v_0,v_0)\mid^2}{16(v_-,\overline{v_-})^2} + \frac{(v_0,\overline{v_0})}{4(v_-,\overline{v_-})} + \frac{1}{4}\Big) \end{aligned}$$

[1]The intervention of the reality of $\tau(\alpha)L^\alpha$ is deliberately left to the next stage.

and

$$\begin{aligned} X_+ &= A\Big(\frac{(v_0,\overline{v_0})}{2\overline{(v_0,v_0)}} - \frac{(v_-,\overline{v_-})}{2\overline{(v_0,v_0)}} - \frac{(v_0,v_0)}{8(v_-,\overline{v_-})}\Big) + \frac{(v_-,\overline{v_-})}{\overline{(v_0,v_0)}} \\ &\quad - \frac{(v_0,v_0)}{2(v_-,\overline{v_-})} + \frac{(v_0,v_0)}{4(v_-,\overline{v_-})} - A^{-1}\Big(\frac{(v_0,\overline{v_0})}{2\overline{(v_0,v_0)}} + \frac{(v_0,v_0)}{8(v_-,\overline{v_-})} + \frac{(v_-,\overline{v_-})}{2\overline{(v_0,v_0)}}\Big); \end{aligned}$$

and, therefore,

$$\begin{aligned} (\rho_V q(\alpha)l^\alpha, q(\alpha)l^\alpha) &= A^2\frac{(v_0,\overline{v_0})^2 - 2(v_-,\overline{v_-})(v_0,\overline{v_0}) + (v_-,\overline{v_-})^2}{2\overline{(v_0,v_0)}} + A^2\frac{(v_0,v_0)}{4} \\ &\quad - A^2\frac{(v_0,\overline{v_0})(v_0,v_0)}{4(v_-,\overline{v_-})} + A^2\frac{(v_0,v_0)^2\overline{(v_0,v_0)}}{32(v_-,\overline{v_-})^2} \\ &\quad - \frac{(v_0,\overline{v_0})^2 + (v_-,\overline{v_-})^2}{\overline{(v_0,v_0)}} + \frac{3(v_0,v_0)}{2} - \frac{(v_0,v_0)^2\overline{(v_0,v_0)}}{16(v_-,\overline{v_-})^2} \\ &\quad + A^{-2}\frac{(v_0,\overline{v_0})^2 + 2(v_0,\overline{v_0})(v_-,\overline{v_-}) + (v_-,\overline{v_-})^2}{2\overline{(v_0,v_0)}} + \\ &\quad + A^{-2}\frac{(v_0,v_0)}{4} + A^{-2}\frac{(v_0,v_0)(v_0,\overline{v_0})}{4(v_-,\overline{v_-})} + A^{-2}\frac{(v_0,v_0)^2\overline{(v_0,v_0)}}{32(v_-,\overline{v_-})^2}; \end{aligned}$$

with

$$A := \frac{\alpha - \overline{\alpha}^{\,-1}}{\alpha + \overline{\alpha}^{\,-1}} = \frac{|\,\alpha\,|^2 - 1}{|\,\alpha\,|^2 + 1} \in \mathbb{R}.$$

On the other hand, in view of the reality of $\tau(\alpha)l^\alpha$, or, equivalently, of the fact that v_0 is a purely imaginary unit, together with equation (B.1), the coefficients X_0, X_- and X_+ simplify to

$$X_0 + 1 = A\Big(\frac{1}{2}\,|\,\alpha\,|^{-1} + \frac{1}{2}\,|\,\alpha\,|\Big)^2 - A^{-1}\Big(\frac{1}{2}\,|\,\alpha\,|^{-1} - \frac{1}{2}\,|\,\alpha\,|\Big)^2,$$

$$X_- = A\Big(\frac{1}{2}\,|\,\alpha\,|^2 + \frac{1}{2}\Big)^2 + A^{-1}\Big(\frac{1}{2}\,|\,\alpha\,|^2 - \frac{1}{2}\Big)^2$$

and

$$X_+ = A\Big(\frac{1}{2}\,|\,\alpha\,|^{-1} + \frac{1}{2}\,|\,\alpha\,|\Big)^2 + A^{-1}\Big(\frac{1}{2}\,|\,\alpha\,| - \frac{1}{2}\,|\,\alpha\,|^{-1}\Big)^2,$$

respectively. In that case,

$$(v_0,v_0)^{-1}(\rho_V q(\alpha)l^\alpha, q(\alpha)l^\alpha) = (X_0+1)^2 + |\,\alpha\,|^{-2}\,X_-X_+,$$

whilst the reality of A ensures the reality of X_0, as well as the positiveness of X_-X_+, leading us to conclude that

$$(\rho_V q(\alpha)l^\alpha, q(\alpha)l^\alpha) \neq 0.$$

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